Consider the following sentence:

This sentence is false.

Is the sentence in the box true or false?

The Liar paradox

If it is cheating to have a sentence refer to itself, then consider this variation.

The sentence in the box is false.

Or this one on the next slide:

The Liar paradox

This sentence on slide 3 of the presentation given in the first class of Paradox and The Philosophical Computer is false.

The Liar paradox

The previous sentence seems to refer to itself only indirectly, and contingently. If I change the order of the slides, the sentence might become true.

The Liar paradox

Another variation:

This sentence is false on Tuesdays.

or

If today is Tuesday, then this sentence is false.

A couple more to consider, especially if you don't like self-reference.

The Liar paradox

This sentence has 47 characters, not including spaces.

The Liar paradox

The previous sentence appears to be true, even though it is self-referential. So self-referring sentences don't seem to always be a problem. At least some self-referential sentences appear to be true. Perhaps including the previous sentence?

The Liar paradox

One last fun one:

The sentence in the next box is false.

The sentence in the previous box is false.

Neither sentence refers to itself, but it creates essentially the same paradox. Perhaps the system as a whole refers to itself? For the previous slide, is it possible that the sentence in the first box is true but the sentence in the second box is false?

Another famous example is: The barber in a certain town is the "one who shaves all those, and those only, who do not shave themselves". The question is, does the barber shave himself? (Adapted from wikipedia.)

And another: The set of all sets that do not contain themselves. (For example, the set of all sets with more than 1 element seems to contain itself, while the set of integers does not contain itself as a member.). The set of all sets that do not contain themselves seems to contain itself if and only if it does not contain itself.

The Liar paradox

Going back to this version and see why it is paradoxical in more detail.

The boxed sentence is false.

The Liar paradox

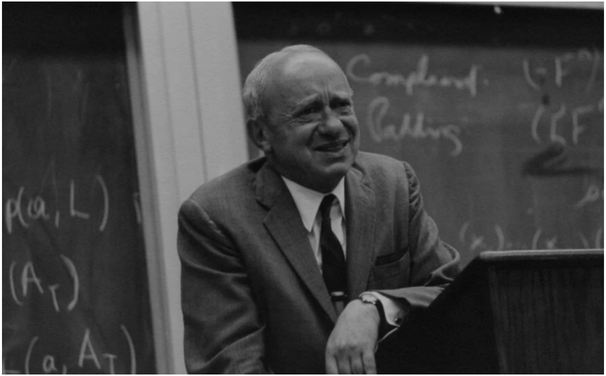
One theory of truth, developed by the logician Alfred Tarski, says the following:

A sentence **‘p’** is true if and only if **p**.

The famous example is

‘Snow is white’ if and only if snow is white.

Tarski, image from LinkedIn



The Liar paradox

According to this idea, let the boxed sentence mean the sentence 'This sentence is false.' Then

'The boxed sentence is false' is true if and only if the boxed sentence is true

Then the boxed sentence is false if and only if the boxed sentence is true.

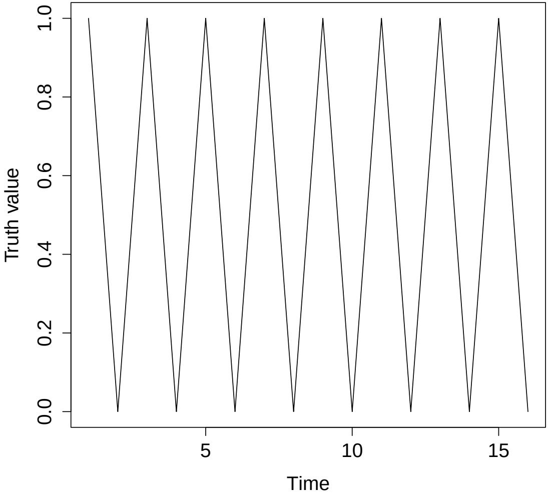
The Liar paradox

Rather than try to resolve the paradox, we'll try to visualize it.

First assume that the box sentenence is true. If it is true, then the sentence asserts that it is false. If we assume it is false, then the sentence says it is not false, so must be true. If it is true, then it is false. etc. This seems to lead to an alternating series of truth values that we consider:

true, false, true, false, true, false, ...

We might visualize this as follows:



The Liar paradox

The previous graph illustrated what might be called *semantic dynamics*, where *semantics* here refers to the interpretation of a sentence as true or false, and dynamics indicates values changing over time.

Things get more interesting if we allow not just two truth values. In *infinite-value logic*, we allow truth values to range continuously from 0 to 1. The idea is that some sentence might be approximately true or closer to truth than others, so we can quantify truth values.

Infinite-valued logic

Some motivations for this include sentences like

- Tomorrow it will rain 1.0 inches.
- Albuquerque has nice weather.
- Albuquerque is sunny.
- Albuquerque is in a desert.
- New York is in a desert.
- Singapore is in a desert.
- Sante Fe is north of Albuqurerque (35°6'39"N 106°35'36"W vs 35°40'2"N 105°57'52"W, from Wikipedia)
- $\pi = 3.14159$.
- $\pi = 3.14$.
- $\pi = 3$.

Infinite-valued logic

The idea is that in bivalent logic (meaning only two truth values are possible), if we assign a sentence to have a truth value of 1, then it's negation has a truth value of 0. Here 1 represents true, and 0 represents false. The negation of a sentence has a truth value of 1 minus the truth value of the original sentence. We generalize this idea in infinite-valued logic.
We can use slashes / to indicate the truth value of a sentence. For example, if **p** is a true sentence then

$$/\mathbf{p}/ = 1$$

. If **p** is false, then we can write

$$/\mathbf{p}/ = 0$$

.
Also we use $\sim$ **p** to mean not-**p**:

$$/\sim \mathbf{p}/ = 1 - /\mathbf{p}/$$

Infinite-valued logic

We assume the rule is still

$$/\sim \mathbf{p}/ = 1 - /\mathbf{p}/$$

including when $0 < /\mathbf{p}/ < 1$.
Logical systems based on this idea include the Kleene and Łukasiewicz systems for infinte-valued logics. Here are some rules for the system.

Infinite-valued logic

The truth value of a conjuction (‘and’ statement) is the minimum of the two truth values.

$$/\mathbf{p}\ \&\ \mathbf{q}/ = \min(/\mathbf{p}/, /\mathbf{q}/)$$

The truth value of a disjunction (‘or’ statement) is the maximum of the two values.

$$/\mathbf{p} \vee \mathbf{q}/ = \max(/\mathbf{p}/, /\mathbf{q}/)$$

Infinite-valued logic

For conditional statements (if **p** then **q**), the Kleene and Łukasiewicz systems give different answers. In classical bivalent logic, however, the statement 'if **p** then **q**' is considered true unless **p** is true and **q** is false. The Kleene approach preserves this with the formula

$$/\mathbf{p} \rightarrow \mathbf{q}/ = \max(1 - /\mathbf{p}/, /\mathbf{q}/) = / \sim \mathbf{p} \vee \mathbf{q}/$$

As an example, suppose $/\mathbf{p}/ = .7$ and $/\mathbf{q}/ = .6$. Then

$$/\mathbf{p} \rightarrow \mathbf{q}/ = \max(1 - /\mathbf{p}/, /\mathbf{q}/) = \max(1 - .7, .6) = \max(.3, .6) = 0.6$$

This can be considered a measure of how easy it is for **p** to be true and **q** to be false.

The material conditional

In traditional symbolic logic, the material conditional is a statement of the form ‘if **p** then **q**’, often written as

$$\mathbf{p} \to \mathbf{q}$$

Some examples of true statements of this form are

- If $x < 5$, then $x < 6$
- if it’s a square, then it’s a rectangle
- if it’s a rectangle, then it’s a square (why is this false?)

The material conditional

Now consider the statement

If *a* is a prime number, then *b* is a prime number

Suppose the statement is false. How would show it is false? You would want to find a case where *a* is prime and *b* is not prime.
This motivates defining the material conditional as being false whenever the antecedant (first phrase) is true, and the consequent (second phrase) is true, and otherwise saying that the statement is true. It leads to some seemingly nonsensical statements as counting as true, however:

- If 0 is odd, then 1 is even.
- If 0 is odd, then 1 is odd.
- If there are over 100 people in this room, then Mars is bigger than Earth.

Sometimes these kinds of examples are called the paradoxes of the material conditional.

The material conditional

The material conditional

$$\mathbf{p} \rightarrow \mathbf{q}$$

is false only when **p** is true and **q** is false. The statement '

$$\sim \mathbf{p} \vee \mathbf{q}$$

' is also false only **p** is true and **q** is false. Thus, in classical symbolic logic, the two statements are exactly equivalent. As they are in the Kleene system for infinite-valued logic.

The material conditional

In the Łukasiewicz system, however, the conditional gets the following formula

$$/\mathbf{p} \rightarrow \mathbf{q}/ = \min(1, 1 - /\mathbf{p}/ + /\mathbf{q}/)$$

or

$$/\mathbf{p} \rightarrow \mathbf{q}/ = \begin{cases} 1 & \text{if } /\mathbf{p}/ \geq /\mathbf{q}/ \\ 1 - /\mathbf{p}/ + /\mathbf{q}/ & \text{if } /\mathbf{p}/ > /\mathbf{q}/ \end{cases}$$

For the previous example of $/\mathbf{p}/ = .7$ and $/\mathbf{q}/ = .6$, the Łukasiewicz conditional gives a value of 1-.7+.6 $= 0.9$, which is quite different from the Kleene answer.

The biconditional

In classical logic, the biconditional is a statement of the form

$$\mathbf{p} \leftrightarrow \mathbf{q}$$

and is equivalent to

$$(\mathbf{p} \to \mathbf{q})\ \&\ (\mathbf{q} \to \mathbf{p})$$

In the Łukasiewicz system, we get

$$/\mathbf{p} \leftrightarrow \mathbf{q}/ = 1 - \mathrm{Abs}(/\mathbf{p}/ - /\mathbf{q}/)$$

In the Kleene system, the biconditional is a bit more complicated (try it!). What values do you get for the two systems for

$$/\mathbf{p} \leftrightarrow \mathbf{q}/$$

when $\mathbf{p} = .6$ and $\mathbf{q} = .7$?

The biconditional

You can think of

$$/\mathbf{p} \leftrightarrow \mathbf{q}/$$

as measuring how similar the two statements are in terms of their truth values.

Infinite-valued logic

Now we'll start considering examples of sentences. Suppose $/\mathbf{p}/$ is the statement that **Albuquerque is sunny**. It is difficult to quantify how true this is. Maybe you could rank a bunch of cities by 'sunniness', maybe days that are less than 'mostly cloudy', and see how Albuquerque compares to other cities.

Let's say we give this sentence a truth value of 0.8. Then

$$/\mathbf{p}/ = 0.8$$

Infinite-valued logic

Now consider the sentence

It is completely true that **Albuquerque is sunny**.

This is really a sentence about another sentence, so a second-order statement. We could think of this way–the second order statement is claiming that $/\mathbf{p}/ = 1$. Since $/\mathbf{p}/ = 0.8$, the claim that $/\mathbf{p}/ = 1$ is false, but not false by a lot. It false by 0.2, which means that its degree of truth is 0.8. So we'll say the second-order statement also has a truth value of 0.8. In other words, adding "It is completely true that" before a statement doesn't change anything.

Infinite-valued logic

Now consider the statement

It is half true that **Albuquerque is sunny**.

This seems to be saying that $/\mathbf{p}/ = 0.5$, when actually $/\mathbf{p}/ = 0.8$. So how true is this seocnd-order statement. The statement would be correct if $/\mathbf{p}/ = 0.5$, and the further away $/\mathbf{p}/$ is from 0.5, the less true the second-order statement seems to be.

So we might say that the second-order statement is wrong by an amount measured by the difference between $/\mathbf{p}/$ and 0.5. Thus the amount wrong is Abs($/\mathbf{p}/$ -0.5) = Abs(0.8-0.5) = 03. Therefore the second order statement has a truth value of 0.7.

The second-order statement has a slightly lower truth value than $/\mathbf{p}/ = 0.8$.

Tarskian schema

The Tarskian schema in bivalent logic says

$$\text{‘}\mathbf{p}\text{’ is true if and only } \mathbf{p}$$

To generalize this to the infinite-valued case, let V**tp** mean the second order statement that **p** has a truth value of **t**. Another notation could be V(**p**)= **t**.

The Tarskian schema can then be written as

$$\mathrm{V}\mathbf{tp} \leftrightarrow \mathbf{p}$$

We can also think of **t** as representing a statement that is always true, like $1 = 1$. In that case, we can think of the truth value of V**tp** as the difference between **t** and **p**.

$$/\mathrm{V}\mathbf{tp}/ = 1 - \mathrm{Abs}(\mathbf{t}\ \text{-}\ \mathbf{/p/})$$

This might seem a little redundant because here $\mathbf{t} = 1$ if we are working with classical logic, but this approach generalizes to other values.

Infinite-valued logic

Now suppose we let V**vp** be the second-order statement that **p** has truth value **v**, where **v** is potentially something between 0 and 1. Then Here we can think of the 1 as represent truth, and Abs(**v - /p/)** as representing a measure of the error.

$$/V\mathbf{vp}/ = 1 - \text{Abs}(\mathbf{v}\text{ - }\mathbf{/p/})$$

We can use this framework to generate sequences of estimates of trruth and falsity like in the Liar paradox.

Recall

The boxed sentence is false.

Call the boxed sentence **b**. Suppose we estimate that $/\mathbf{b}/ = 0.25$. We'll let **f** represent a truth value of 0. Then

$$/\mathrm{V}\mathbf{fb}/ = 1 \text{ - Abs(0-1/4) } = 1 - 1/4 = 3/4.$$

We can think of estimates of the truth of the Liar statement (the boxed statement) as having a sequence of truth values $x_0, x_1, \ldots$ where $x_0 = 1/4$ was the initial estimate.

The Simple Liar in Infinite-valued logic

Now we represent the truth value as a recursion:

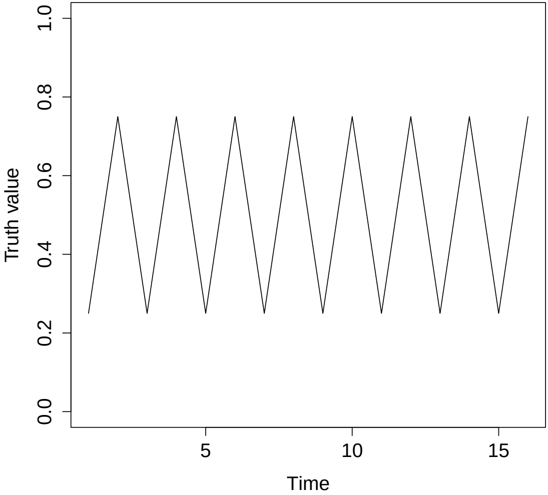
$$x_{n+1} = 1 - \text{Abs}(0 - x_n)$$

where x_0 is the initial estimate, in this case $1/4$. This gives

$$\begin{aligned}
x_0 &= 1/4 \\
x_1 &= 1 - \text{Abs}(0 - x_0) = 1 - \text{Abs}(0 - 1/4) = 3/4 \\
x_2 &= 1 - \text{Abs}(0 - x_1) = 1 - \text{Abs}(0 - 3/4) = 1/4 \\
x_3 &= 1 - \text{Abs}(0 - x_2) = 1 - \text{Abs}(0 - 1/4) = 3/4
\end{aligned}$$

The system oscillates between $1/4$ and $3/4$. This is like saying that if the boxed sentence is mostly false, then it is mostly true, but if it is mostly true, then it is mostly false, and so forever.

Plotting this as a time series, we get



Some weird sentences

We can imagine sentences that make claims about their estimated truth values instead of their actual truth values. Here is an example, called the *Half-Sayer*:

This sentence is as true as half its estimated value.

In terms of the V**vp** schema, the successive truth values are

$$x_{n+1} = 1 - \text{Abs}(0.5x_n - x_n)$$

Here are some values we get if we start with $x_0 = 0.5$.

Term	value
x_0	0.5
x_1	0.75
x_2	1-abs(.5*.75-.75) $=$ 0.625
x_3	1-abs(.5*.625-.625) $=$ 0.6875

If you continue this sequence, it seems to converge to $2/3$. It's useful to write code to figure this out.

Another weird sentence is the following, called the Minimalist.

This sentence is as true as whichever is smaller: its estimated value or the op

This translates to the rule

$$x_{n+1} = 1 - \text{Abs}(\min(x_n, 1 - x_n) - x_n)$$

So if we start with $x_0 = 0.6$, we get

$$\begin{aligned}
x_1 &= 1 - \text{Abs}(\min(.6, .4) - .6) = 1 - .2 = .8 \\
x_2 &= 1 - \text{Abs}(\min(.8, .2) - .8) = .4 \\
x_3 &= 1 - \text{Abs}(\min(.4, .6) - .4) = 1 \\
x_4 &= 1 - \text{Abs}(\min(1, 0) - 1) = 0 \\
x_5 &= 0
\end{aligned}$$

So you get something unstable that alternates between 0 and 1.

For the minimalist, if we start with an estimated truth value, we get

$$x_0 = 2/3$$
$$x_1 = 1 - \text{Abs}(\min(2/3.1/3) - 2/3) = 2/3$$

So 2/3 is a fixed point–it leads to a stable truth value, but it is an unstable equilibrium point. Any sttarting value away from 2/3 leads to oscillating between 0 and 1.

Now consider the sentence

This sentence is true

If we start with the assumption that it is true, then it seems to be stable. We could also investigate what happens with other starting values. Using the schema, we have

$$x_n = 1 - \text{Abs}(1 - x_n)$$

Thus, if $x_0 = 1$, then $x_1 = 1 - 0 = 1$, so this is a solution. However, if $x_0 = .1$, then $x_1 = 1 - \text{Abs}(1 - .1) = 1 - .9 = .1$. Thus, assuming that the statement has a truth value of 0.1 is also stable. There doesn't seem to be a way of deciding what the truth value is. The statement is compatible with any truth value, including 0.

Introduction to R

A goal for this week is for you to be able to learn a little bit of R, especially to use it write your own functions, and to make some interesting plots.

R is free to download. You can download a standalone version here `https://www.r-project.org/` Or you can download it as part of R Studio, an evironment for running R.
`https://posit.co/download/rstudio-desktop/`

It is also possible to use it online without downloading and installing it.
`https://rdrr.io/snippets/`

R examples

R can be used as a calculator with standard scientific functions, including things like binomial coefficients, factorials, and trig functions.

```
> (pi-3.14)/pi # relative error
[1] 0.0005069574
> (pi-3)/pi # relative error
[1] 0.04507034
R is vector oriented, so objects are often treated as vectors
> x <- 2:11
> x
 [1]  2  3  4  5  6  7  8  9 10 11
> y <- sample(1:6,10,replace=TRUE)
> y
 [1] 1 1 1 4 5 3 2 4 4 2
> x*y
 [1]  2  3  4 20 30 21 16 36 40 22
```

R examples

```
> choose(5,3) # binomial coefficient
[1] 10
> choose(52,5) # number of poker hands
[1] 2598960
> cosh(sqrt(pi)) # hyperbolic cosine
[1] 3.027596
>  atan(.2) # arctan or inverse tangent
[1] 0.1973956
>  runif(10,3,7) # random numbers between 3 and 7
 [1] 3.155183 3.972129 4.062079 3.957624 4.107803 3.525488 5.353248
 [9] 4.456439 3.125219
```

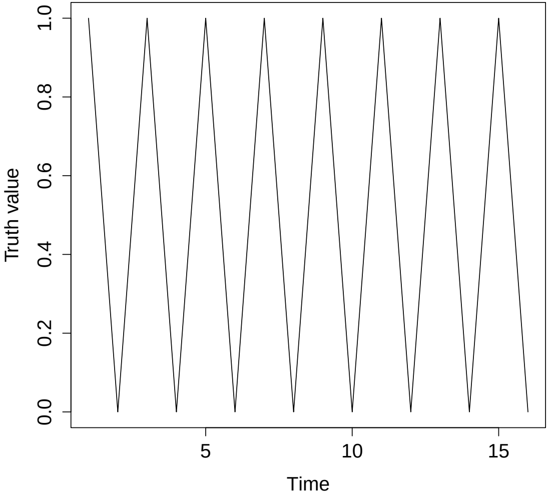
R examples

```
> z <- rep(3,5) # rep operator repeats a value a given number of times
> z
[1] 3 3 3 3 3
> w <- rep(4,2)
> w
[1] 4 4
> v <- c(w,z) # concatenate operator allows combining two vectors
> v
[1] 4 4 3 3 3 3 3
```

R examples

```
> y <- rep(c(1,0),8)
> y
[1] 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0
> x <- 1:16
> #below, cex.axis and cex.lab affect the font size
> #type="l" means I want a line graph instead of individual points.
> plot(x,y,type="l",cex.axis=1.4,cex.lab=1.4,ylab="Truth value",
  xlab="Time")
> pdf("Liar1.pdf") #opens file to write plot to instead of screen
> plot(x,y,type="l",cex.axis=1.4,cex.lab=1.4,ylab="Truth value",
  xlab="Time")
> dev.off() #closes file
```

The classical liar over time



Writing your own functions

The following code implements the function

$$f(n) = 2n^2 + 3$$

```
myfunction <- function(n) {
value <- 2*n^2+3
return(value)
}
> myfunction(4)
[1] 35
```

Writing your own functions

For the Half Sayer truth example

```
 > truth <- function(x) {
   value <- 1-abs(.5*x-x)
   return(value)
 }
 > truth(.5)
[1] 0.75
 > truth(truth(.5))
[1] 0.625
> truth(truth(truth(.5)))
[1] 0.6875
> truth(truth(truth(truth(.5))))
[1] 0.65625
> truth(truth(truth(truth(truth(.5)))))
[1] 0.671875
> truth(2/3)
[1] 0.6666667
```

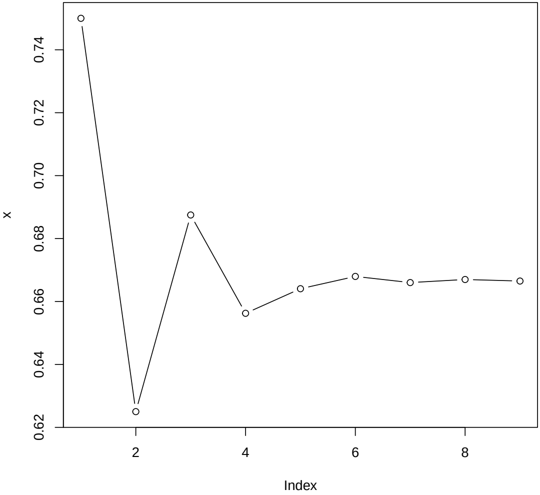
In the previous example, $2/3$ is a fixed point, and it can be considered a "solution" to the strange statement that this is statement is half as true as its estimated value. So if the estimated value is $2/3$, the statement says that it's truth value is half of that, which is $1/3$. This not true, but also not completely false (think of $1/3$ as an estimate of $2/3$), and we think of the correctness of the statement as 1 minus the difference in estimated values. So the statement seems to be $2/3$ true in infinite-valued logic.

Writing your own functions

For the Half Sayer truth example, we might want code that will apply the recursion many times. This assumes that the `truth()` function is already defined.

```
 truth2 <- function(x,maxterms) {
   temp <- truth(x)
   for(i in 1:maxterms) {
      print(temp)
      temp <- truth(temp)
   }
}
> truth2(.5,10)
[1] 0.75
[1] 0.625
[1] 0.6875
[1] 0.65625
[1] 0.671875
[1] 0.6640625
[1] 0.6679688
[1] 0.6660156
[1] 0.6669922
[1] 0.6665039
```

Time plot of the Half Sayer, starting with $x_0 = 0.75$.



Interpreting very and fairly

In the field of *fuzzy logic*, it is standard to interpret the terms 'very' and 'fairly' in terms of squaring and square roots. For example, let **p** be the statement **Paul is tall**. and let **q** be the statement **Paul is good at tennis**.

Assume that $/\mathbf{p}/ = 0.9$ and $/\mathbf{q}/ = 0.09$. Then we might assign the statement **Paul is very tall** a truth value of $0.9^2 = 0.81$ and **Paul is fairly good at tennis** a truth value of $\sqrt{.09} = 0.3$. The idea is that if someone is claimed to be very tall instead of just tall, we have a higher expectation of height. If someone is claimed to be fairly good at something we have a somewhat lower expectation than if was just an unqualified claim of being good.

Two more weird sentences

Modest Liar: This sentence is fairly false

Emphatic Liar: This sentence is very false

Emphatic Liar

For the emphatic liar setence, we can use the schema used for the simple liar and square it to reflect using the word ‘very’:

$$x_{n+1} = (1 - \mathrm{Abs}(0 - x_n))^2 \Rightarrow x_{n+1} = (1 - x_n)^2$$

For the Modest liar, we get

$$x_{n+1} = \sqrt{1 - \mathrm{Abs}(0 - x_n)} \Rightarrow x_{n+1} = \sqrt{1 - x_n}$$

Weird sentences in R: Emphatic Liar

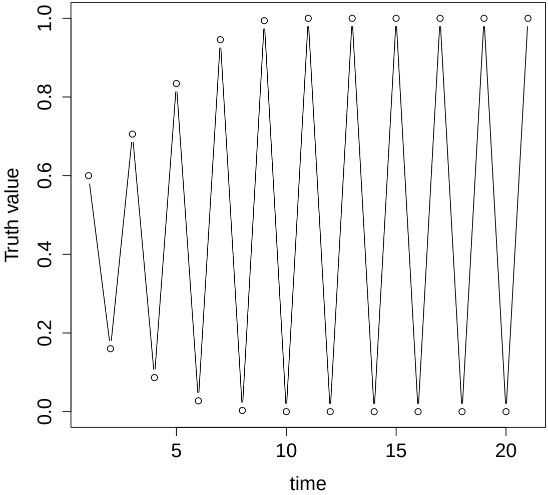
Let's write functions R to figure out what these sentences do.

```
emphatic <- function(x) {
  value <- (1-x)^2
  return(value)
}
emphatic2 <- function(x,maxterms) {
  value <- rep(-1,maxterms)
  temp <- emphatic(x)
  for(i in 1:maxterms) {
    print(temp)
    value[i] <- temp
    temp <- emphatic(temp)
  }
  return(value)
}
```

```
> y <- emphatic2(.6,20)
[1] 0.16
[1] 0.7056
[1] 0.08667136
[1] 0.8341692
[1] 0.02749985
[1] 0.9457565
[1] 0.002942353
[1] 0.994124
[1] 3.452795e-05
[1] 0.9999309
[1] 4.768554e-09
[1] 1
[1] 9.095642e-17
[1] 1
[1] 4.930381e-32
[1] 1
[1] 0
[1] 1
[1] 0
[1] 1
```

```
> y <- emphatict2(.6,20)
> y
[1] 1.600000e-01 7.056000e-01 8.667136e-02 8.341692e-01 2.749985e-0
[6] 9.457565e-01 2.942353e-03 9.941240e-01 3.452795e-05 9.999309e-0
[11] 4.768554e-09 1.000000e+00 9.095642e-17 1.000000e+00 4.930381e-
[16] 1.000000e+00 0.000000e+00 1.000000e+00 0.000000e+00 1.000000e+
```

Time plot of the Emphatic Liar starting with $x_0 = 0.6$.



Similar to an example from yesterday, this sentence has an unstable equilibrium point. The equilibrium point is at $x = (3 - \sqrt{5})/2$, which is related to the Golden Ratio.

```
x <- (3-sqrt(5))/2
> x
[1] 0.381966
> emphatic2(x,6)
[1] 0.381966
[1] 0.381966
[1] 0.381966
[1] 0.381966
[1] 0.381966
[1] 0.381966
```

Relationship to the Golden Ratio

The Golden Ratio was studied a lot by the Greek artist Phidias, and gets a Greek letter based on his name:

$$\phi = \frac{1+\sqrt{5}}{2}$$

To see the relationship consider

$$x = 1 - 1/\phi$$

You can show that $x = (3-\sqrt{5})/2$

The modest liar

The modest liar is the sentence

This sentence is fairly false

with function

$$x_{n+1} = \sqrt{(1 - x_n)}$$

We'll now look at R code for it

The modest liar

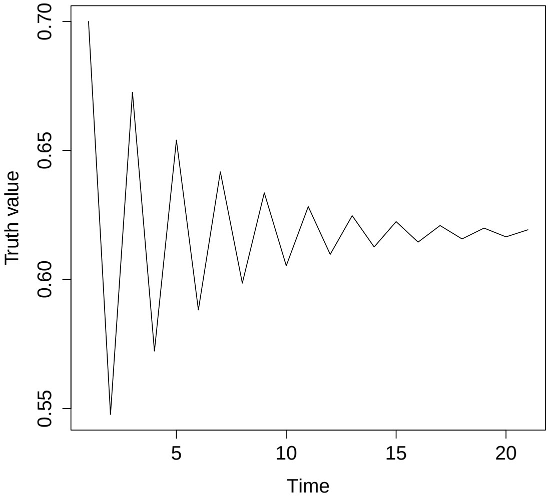
```
modest <- function(x) {
  value <- sqrt(1-x)
  return(value)
}
modest2 <- function(x,maxterms) {
  value <- rep(-1,maxterms)
  temp <- modest(x)
  for(i in 1:maxterms) {
    print(temp)
    value[i] <- temp
    temp <- modest(temp)
  }
  return(value)
}
```

Modest iiar

```
y <- modest2(.7,20)
y
[1] 0.5477226
[1] 0.6725158
[1] 0.5722624
[1] 0.6540165
[1] 0.5882036
[1] 0.6417136
[1] 0.5985703
[1] 0.6335848
[1] 0.6053224
[1] 0.6282337
[1] 0.6097264
[1] 0.6247188
[1] 0.612602
[1] 0.6224131
[1] 0.614481
[1] 0.6209017
[1] 0.6157096
[1] 0.6199116
[1] 0.6165131
[1] 0.6192632
```

Time plot of the Modest Liar starting with $x_0 = 0.7$.

```
> getwd()
[1] "/Users/jadcassvc/Dropbox/Mac/Documents/Zoom"
> setwd("/Users/jadcassvc/Dropbox/Mac/Documents/Teaching/2024-Campersand")
> pdf("Paradox5.pdf")
> plot(1:21,c(.7,y),xlab="Time",ylab="Truth value",cex.axis=1.4,cex.lab=1.4,type="l")
> dev.off()
```


0.70
0.65
0.60
0.55
Truth value
5
10
15
20
Time

Modest liar

The modest liar is fairly stable, wtih a truth value approaching $\phi - 1 \approx 0.618034$ regardless of the initial value. as long as $0 < x_0 < 1$. If you start at 0 or 1, then it oscillates.

```
y <- modest2(.99,20)
y
 [1] 0.1000000 0.9486833 0.2265319 0.8794704 0.3471738 0.8079766 0.4382047
 [8] 0.7495300 0.5004698 0.7067745 0.5415030 0.6771241 0.5682217 0.6570984
[15] 0.5855780 0.6437562 0.5968617 0.6349318 0.6042088 0.6291194
y <- modest2(1,20)
y
[1] 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
```

The chaotic liar

Consider this sentence

The statement is as true as it is estimated to be false.

So if you estimate the boxed sentence to have a truth value of 0.3, the sentence is saying that it's truth value is $1 - 0.3 = 0.7$.

We can express the truth value sequence as

$$x_{n+1} = 1 - \text{Abs}((1 - x_n) - x_n)$$

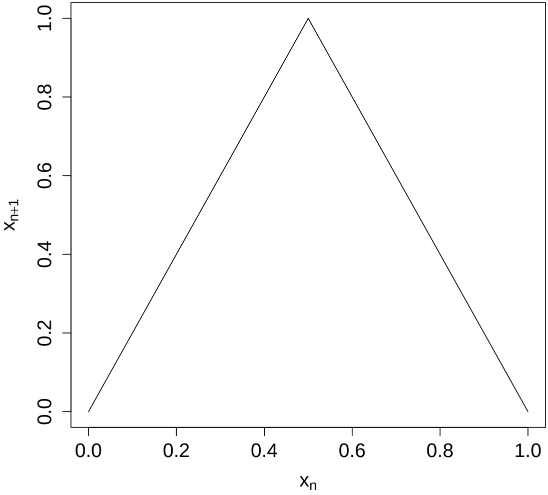
To parse this, x_n represents an estimate of how true the sentence is, and $(1 - x_n)$ measures how much the sentence is estimated to be false. So the absolute value of the difference between these is an estimate of how false the chaotic liar sentence is after estimating x_n. Subtracting from 1 gives how true the sentence appears to be.

The chaotic liar

The function can also be written as

$$x_{n+1} = \begin{cases} 2x_n & \text{for } 0 \leq x \leq 1/2 \\ 2(1 - x_n) & \text{for } 1/2 < x \leq 1 \end{cases}$$

A plot of the chaotic liar function.



The chaotic liar

Let's try some examples by hand before trying in the computer.
Suppose the initial estimate is $x_0 = 0.3$. Then

$$\begin{aligned}
x_1 &= 1 - \text{Abs}((1 - x_0) - x_0) = 1 - \text{Abs}((1 - 0.3) - 0.3) \\
&= 1 - \text{Abs}(0.7 - 0.3) = 1 - 0.4 = 0.6 \\
x_2 &= 1 - \text{Abs}(1 - 0.6) - 0.6) = 1 - \text{Abs}(0.4 - 0.6) = 1 - 0.2 = 0.8 \\
x_3 &= 1 - \text{Abs}(1 - 0.8) - 0.8) = 1 - \text{Abs}(0.2 - 0.8) = 0.4 \\
x_4 &= 1 - \text{Abs}(1 - 0.4) - 0.4) = 1 - \text{Abs}(0.6 - 0.4) = 0.8
\end{aligned}$$

At this point, the series will alternate between 0.4 and 0.8

The chaotic liar

Now let's try a slightly different starting point, say $x = 0.31$.

$$
\begin{aligned}
x_1 &= 1 - \text{Abs}((1 - x_0) - x_0) = 1 - \text{Abs}((1 - 0.31) - 0.31) \\
&= 1 - \text{Abs}(0.69 - 0.31) = 1 - 0.38 = 0.62 \\
x_2 &= 1 - \text{Abs}(1 - 0.62) - 0.62) = 1 - \text{Abs}(0.38 - 0.62) = 1 - 0.24 = 0.76 \\
x_3 &= 1 - \text{Abs}(1 - 0.76) - 0.76) = 1 - \text{Abs}(0.24 - 0.76) = 0.48 \\
x_4 &= 1 - \text{Abs}(1 - 0.52) - 0.52) = 1 - \text{Abs}(0.48 - 0.52) = 0.96 \\
x_5 &= .08 \\
x_6 &= 0.16
\end{aligned}
$$

So far, the series is not repeating. Is it possible for it to go on forever without repeating? Why or why not?

The chaotic liar

There are fixed points at $x_0 = 0$ and $x_0 = 0.5$.

For $x_0 = 0$:

$$\begin{aligned} x_1 &= 1 - \text{Abs}((1 - x_0) - x_0) = 1 - \text{Abs}((1 - 0) - 0) \\ &= 0 \end{aligned}$$

For $x_0 = 2/3$,

$$\begin{aligned} x_1 &= 1 - \text{Abs}((1 - 2/3) - 2/3) = 1 - \text{Abs}(1/3 - 2/3) \\ &= 2/3 \end{aligned}$$

So the sentence is compatible with being just false (truth value 0) or 2/3 true.

The chaotic liar also can result in a value of 0 from a different starting point, but then gets stuck at 0. For example, if $x_0 = 1/2$, then

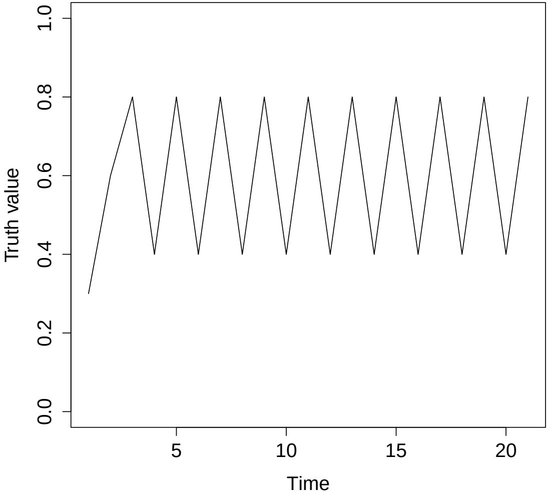
$$
\begin{aligned}
x_1 &= 1 - \text{Abs}((1 - 1/2) - 1/2) = 1 - \text{Abs}(1 - 2 - 1/2) = 1 \\
x_2 &= 1 - \text{Abs}((1 - 1) - 1) = 0 \\
&= 0
\end{aligned}
$$

The chaotic liar in R

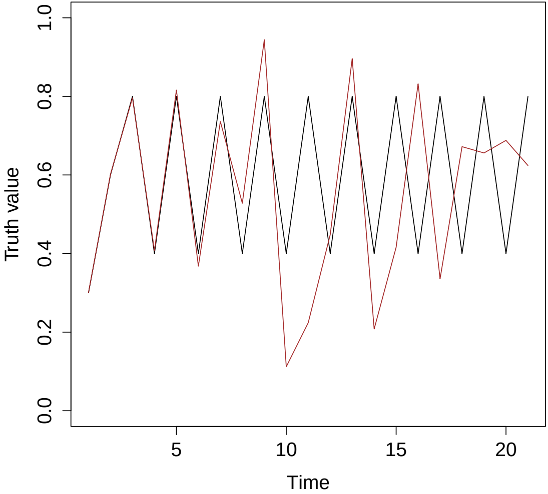
```
chaos <- function(x) {
  value <- 1-abs((1-x)-x)
  return(value)
}
chaos2 <- function(x,maxterms) {
  value <- rep(-1,maxterms)
  temp <- chaos(x)
  for(i in 1:maxterms) {
    print(temp)
    value[i] <- temp
    temp <- chaos(temp)
  }
  return(value)
}
```

```
y300 <- chaos2(.3,20)
y301 <- chaos2(.301,20)
y302 <- chaos2(.302,20)
y303 <- chaos2(.303,20)
x <- 1:21
plot(x,c(.3,y300),type="l",col="black",cex.axis=1.4,cex.lab=1.4,
  xlab="Time",ylab="Truth value",ylim=c(0,1))
points(x,c(.301,y301),type="l",col="brown")
points(x,c(.302,y302),type="l",col="red")
points(x,c(.303,y303),type="l",col="orange")
```

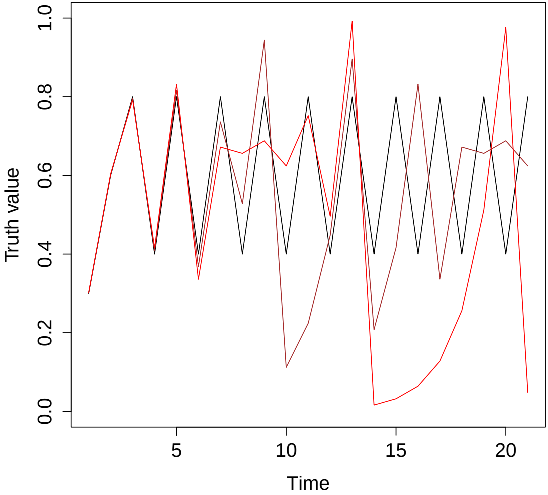
Time plot of the chaotic liar starting with $x_0 = 0.300$.



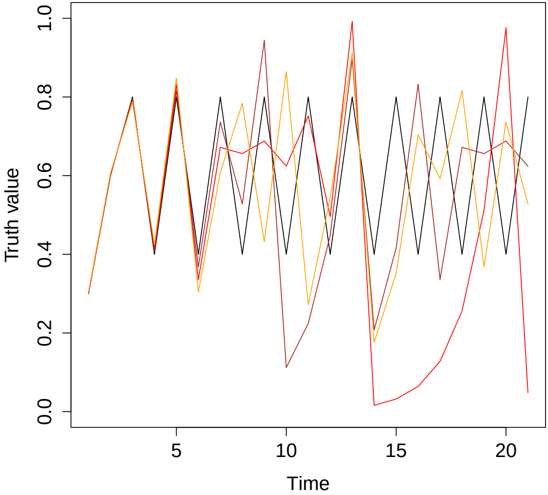
Time plot of the chaotic liar starting with $x_0 = 0.300, 0.301$.



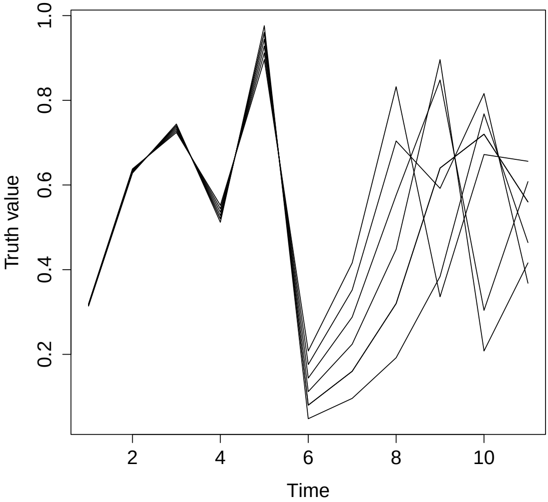
Time plot of the Emphatic Liar starting with $x_0 = 0.300, 0.301, 0.302$.



Time plot of the chaotic liar starting with $x_0 = 0.300, 0.301, 0.302, 0.303$.



Time plot of the chaotic liar starting with $x_0 = 0.314, \ldots, 0.319$.



The chaotic liar

The chaotic gets it's name from the fact that it is extremely sensitive to initial conditions. Previous examples we've seen have either converged to a limiting value for different starting values, or ended up alternating between two values, such as 0 and 1.

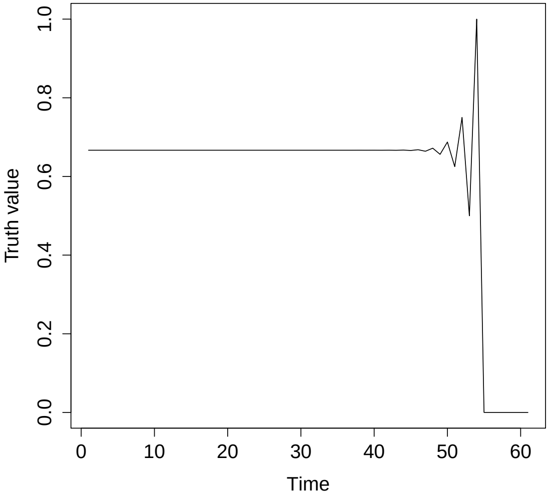
One issue with doing these calculations in a computer is that it is sensitive to round off error. Since for a chaotic system, then dynamics are very sensitive to the initial condition, that means that if there is a small amount of round off error, this can cause the calculations to become inaccurate, and more innaccurate over time.

As an example, consider what R does with $x_0 = 2./3$. We know that this is a fixed point. so it shouldn't change. However, R doesn't use infinite precision, so $2/3$ is represented with some round off error in the computer.

The chaotic liar

```
> y <- chaos2(2/3,60)
> y
 [1] 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667
 [8] 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667
[15] 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667
[22] 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667 0.6666667
[29] 0.6666667 0.6666666 0.6666667 0.6666665 0.6666670 0.6666660 0.6666679
[36] 0.6666641 0.6666718 0.6666565 0.6666870 0.6666260 0.6667480 0.6665039
[43] 0.6669922 0.6660156 0.6679688 0.6640625 0.6718750 0.6562500 0.6875000
[50] 0.6250000 0.7500000 0.5000000 1.0000000 0.0000000 0.0000000 0.0000000
[57] 0.0000000 0.0000000 0.0000000 0.0000000
```

Time plot of the chaotic liar starting with $x_0 = 2/3$ (approximated in R).



The logistic liar

The following are (admittedly confusing) variations on the chaotic liar:

It is false that this statement is as true as it is estimated to be false.

It is very false that this statement is as true as it is estimated to be false.