According to your paper in 1991, we can split the column space $C(I - M_X)$ into three orthogonal parts as the following:

$$\begin{aligned} & I - M_X = (I - M_{XZ}) + (M_Z - M_0) + M_* \\ & \textbf{where} \qquad M_* = (M_{XZ} - M_X) - (M_Z - M_0) \qquad (1) \\ & \textbf{with} \qquad M_0 = M_{M_Z X} \end{aligned}$$

I am curious if the M_* is a ppo. As M_{XZ}, M_X, M_Z, and M_0 are symmetric, what I have to prove is the idempotent property of the M_*. I tried to prove it by the following.

$$\begin{aligned} M_* M_* &= [(M_{XZ} - M_X) - (M_Z - M_0)][(M_{XZ} - M_X) - (M_Z - M_0)] \\ &= (M_{XZ} - M_X)(M_{XZ} - M_X) - (M_{XZ} - M_X)(M_Z - M_0) \\ &\quad - (M_Z - M_0)(M_{XZ} - M_X) + (M_Z - M_0)(M_Z - M_0) \\ &= (M_{XZ} - M_X) - 2(M_Z - M_0 - M_X M_Z + M_X M_0) + (M_Z - M_0) \\ &= (M_{XZ} - M_X) - (M_Z - M_0) + 2M_X(M_Z - M_0) \end{aligned}$$

If M_* is idempotent, $C(M_Z - M_0)$ has to be orthogonal to $C(X)$, i.e. $C(M_Z - M_0) \in C(X)^{\perp}$. The following is my question, does $C(M_Z - M_0) = C(M_Z X)^{\perp}_{C(Z)}$ or $C(M_Z - M_0) \in C(M_Z X)^{\perp}_{C(Z)}$? If either one of the above expression is true, then for any $a \in C(M_Z - M_0)$, we have $a = Z\alpha$ for some α and $(M_Z X)'a = 0$. Since $(M_Z X)'a = 0 \Rightarrow X'M_Z Z\alpha = 0 \Rightarrow X'Z\alpha = 0 \Rightarrow X'a = 0$, we have $M_X(M_Z - M_0) = 0$. Thus, the M_* is a ppo.

But unfortunately, the program I wrote for fitting the data in table 7.5 in the ANOVA book did not suggest the M_* is idempotent. Therefore, I am quite confused with the relationship between $C(M_Z - M_0)$ and $C(M_Z X)^{\perp}_{C(Z)}$.