SEQUENCES OF VECTORS THAT ARE ORBITS OF OPERATORS

Ralph deLaubenfels

Ohio State University

ABSTRACT. This paper characterizes sequences of vectors that are the orbits of a linear operator and sequences of vectors in a Hilbert space that are orbits of a unitary operator. The latter is applied to time series. Sequences of vectors in a Hilbert space that generalize random walks are also shown to be the orbits of a bounded linear operator.

1991 *Mathematics Subject Classification*. Primary 47A99, Secondary 60G10, 60G50.

0. INTRODUCTION. An *orbit* of the operator T is, for some x, $\{T^k x\}_{k=0}^{\infty}$ or, if T is invertible, $\{T^k x\}_{k\in\mathbf{Z}}$.

This paper considers, given a sequence of vectors $\{u_k\}$, the existence of a linear operator U such that $u_k = U^k u_0$ for all k. In other words, we ask, when is a sequence of vectors the orbit of some linear operator U?

Of more interest is to have a sequence be the orbit of a bounded operator U; then the benefits of operator theory can be invoked. For example, one then automatically has well-posedness: small mistakes in estimating u_0 lead to controlled errors in estimates of the entire sequence. In general, producing U could be said to be taking us from local to global behaviour. Perhaps of the most interest is to have U unitary; then we may apply the spectral theorem to obtain extensive information about the original sequence (see Theorem 2.3 and Example 2.5).

Section I is purely algebraic, characterizing sequences of vectors that are orbits of a linear operator. Section II takes place on a Hilbert space, characterizing sequences that are orbits of a unitary operator (Theorem 2.3) and considering sequences that are partial sums from an orthogonal sequence (Theorem 2.6). These theorems are motivated by popular discrete stochastic processes; weakly stationary time series are a special case of Theorem 2.3 (see Example 2.5), and random walks are a special case of Theorem 2.6 (see Example 2.7).

Theorem 2.3 shows that a sequence $\{u_k\}_{k\in\mathbf{Z}}$ in a Hilbert space is the orbit of a unitary operator if and only if $\langle u_n, u_m\rangle = \langle u_{n+k}, u_{m+k}\rangle$ for all integers n, m, k. This is also equivalent to $\{u_k\}_{k\in\mathbf{Z}}$ being the moments of an appropriate vector-valued measure, which in turn is equivalent to $\{\langle u_k, u_0\rangle\}_{k\in\mathbf{Z}}$ being the moments of a positive measure.

This paper does not pretend to present new results about stochastic processes; however, as a special case of Theorem 2.3, we obtain a fresh, simplified, and unified look at stochastic processes, including a very short proof of the spectral representation of a weakly stationary time series (see Example 2.5).

I. ALGEBRAIC RESULTS. In the following two propositions, it should be clear how to replace $\{u_k\}_{k=0}^{\infty}$ with $\{u_k\}_{k=-\infty}^{\infty}$.

Proposition 1.1. *Suppose $\{u_k\}_{k=0}^{\infty} \subseteq X$, a vector space. The following are equivalent.*

(a) *There exists a linear operator U on the span of $\{u_k\}_{k=0}^{\infty}$ such that $U(u_k) = u_{k+1}$, for $k = 0, 1, 2, \ldots$*

(b) *If, for $\alpha_k \in \mathbf{C}$, $\sum_{k=0}^{N} \alpha_k u_k = 0$, then $\sum_{k=0}^{N} \alpha_k u_{k+1} = 0$.*

Proof: (a) $\rightarrow$ (b). $0 = U(0) = U\left(\sum_{k=0}^{N} \alpha_k u_k\right) = \sum_{k=0}^{n} \alpha_k u_{k+1}$.

(b) $\rightarrow$ (a). $U\left(\sum_{k=0}^{N} \alpha_k u_k\right) \equiv \sum_{k=0}^{N} \alpha_k u_{k+1}$, for $\{\alpha_k\}_{k=0}^{N} \subseteq \mathbf{C}$, unambiguously defines U, since, for $N \geq M$,

$$\sum_{k=0}^{N} \alpha_k u_k = \sum_{j=0}^{M} \beta_j u_j$$

implies that (defining $\beta_j = 0$ for $j > M$) $0 = \sum_{k=0}^{N} (\alpha_k - \beta_k) u_k$, so that $0 = \sum_{k=0}^{N} (\alpha_k - \beta_k) u_{k+1}$, hence

$$\sum_{k=0}^{N} \alpha_k u_{k+1} = \sum_{j=0}^{M} \beta_j u_{j+1}.$$

The linearity of U is immediate. $\square$

In particular, in order that a sequence be an orbit of an operator, as in (a) of Proposition 1.1, it is sufficient that the sequence be linearly independent. The following shows that it is almost necessary.

Proposition 1.2. *Suppose $\{u_k\}_{k=0}^{\infty} \subseteq X$, a vector space, and $X_0 \equiv span\{u_k\}_{k=0}^{\infty}$. The following are equivalent.*

(a) *X_0 is infinite dimensional and there exists a linear operator U on X_0 such that $U(u_k) = u_{k+1}$, for $k = 0, 1, 2, \ldots$*

(b) *$\{u_k\}_{k=0}^{\infty}$ is linearly independent.*

Proof: (a) $\to$ (b). Suppose, for the sake of contradiction, there exists a nonnegative integer M, and complex $\{\alpha_k\}_{k=0}^{M-1}$, such that

$$u_M = \sum_{k=0}^{M-1} \alpha_k u_k.$$

Then $u_M = U(u_{M-1}) \in W \equiv \text{span}\{u_0, u_1, ..., u_{M-1}\}$. For $0 \le k < N-1$, $U(u_k) = u_{k+1} \in W$. Thus U maps W into itself. This implies that, for any $n \in \mathbf{N}$,

$$u_n = U^n(u_0) \in W,$$

hence $X_0 \subseteq W$, contradicting the fact that X_0 is infinite dimensional.
(b) $\to$ (a). Proposition 1.1 implies that the desired operator U exists. The infinite dimensionality of X_0 is clear by definition. $\square$

II. ORBITS OF BOUNDED OPERATORS ON A HILBERT SPACE. Throughout this section, $(H, \langle\ \rangle)$ is a complex Hilbert space. Examples given are stochastic processes, which are sequences of vectors in the (complexification of the) Hilbert space of random variables of finite variance with zero mean, inner product

$$\langle X, Y\rangle \equiv \text{Cov}(X, Y).$$

Definition 2.1. We will call the sequence $\{u_k\}_{k\in\mathbf{Z}} \subseteq H$ *unitary* if

$$\langle u_n, u_m\rangle = \langle u_{n+k}, u_{m+k}\rangle\,,$$

for all $n, m, k \in \mathbf{Z}$.

Definition 2.2. Let Z be a countably additive (in the norm of H) H-valued measure defined on the Borel subsets of the complex plane, $x \in H$. We will say that Z is *orthogonal with respect to* x if $Z(\phi) = 0, Z(\mathbf{C}) = x$, and

$$\langle Z(A), Z(B)\rangle = \langle Z(A\cap B), x\rangle\,,$$

for all Borel sets A, B.

Note that, for any Borel A,

$$\|Z(A)\|^2 = \langle Z(A), Z(A)\rangle = \langle Z(A\cap A), x\rangle = \langle Z(A), x\rangle\,,$$

hence $B \mapsto \langle Z(B), x\rangle$ is a positive Borel measure, and

$$\|Z(A)\|^2 \le \langle Z(\mathbf{C}), x\rangle = \|x\|^2$$

for any Borel A. By [Di-U, Proposition I.1.11b], Z is of bounded semivariation (see [Di-U, Proposition I.1.11a]), so that, for any Borel measurable complex-valued function f,

$$\int_{\mathbf{C}} f(z)\, dZ(z)$$

is defined as a limit of integrals of simple functions converging uniformly to f on the support of Z. See [Di-U], especially [Di-U, Theorem II.4.1], for information about vector-valued measures.

In the following, Υ denotes the unit circle in the complex plane.

Theorem 2.3. *Suppose $\{u_k\}_{k\in\mathbf{Z}} \subseteq H$. Then the following are equivalent.*

(a) *$\{u_k\}_{k\in\mathbf{Z}}$ is unitary.*

(b) *There exists a unitary operator U on the closure of the span of $\{u_k\}_{k\in\mathbf{Z}}$ such that $u_k = U^k u_0$, for all $k \in \mathbf{Z}$.*

(c) *There exists a vector-valued measure Z orthogonal with respect to u_0, supported on the unit circle Υ, such that*

$$u_n = \int_{\Upsilon} z^n \, dZ(z),$$

for all integers n.

(d) *There exists a measure μ on the unit circle Υ such that*

$$\langle u_n, u_m \rangle = \int_{\Upsilon} z^{n-m} \, d\mu(z),$$

for all integers n, m.

Proof: (a) $\rightarrow$ (b). For $\alpha_k \in \mathbf{C}$,

$$\|\sum_{k=m}^{n} \alpha_k u_k\|^2 = \left\langle \sum_{k=m}^{n} \alpha_k u_k, \sum_{j=m}^{n} \alpha_j u_j \right\rangle = \sum_{k,j} \alpha_k \overline{\alpha_j} \, \langle u_k, u_j \rangle = \sum_{k,j} \alpha_k \overline{\alpha_j} \, \langle u_{k+1}, u_{j+1} \rangle$$
$$= \|\sum_{k=m}^{n} \alpha_k u_{k+1}\|^2.$$

By Proposition 1.1, there exists an isometry U on the span of $\{u_k\}_{k \in \mathbf{Z}}$ such that $U(u_k) = u_{k+1}$, for all integers k. This implies that $u_k = U^k u_0$, for all integers k, and U is surjective (still on the span of $\{u_k\}_{k \in \mathbf{Z}}$). The operator U extends uniquely, in the usual way, to a surjective isometry, hence a unitary map, on the closure of the span of $\{u_k\}_{k \in \mathbf{Z}}$.
(b) $\rightarrow$ (c). By the spectral theorem (see [Ru, Theorem 12.23]), there exists a self-adjoint projection-valued measure E such that

$$U^k = \int_{\Upsilon} z^k \, dE(z),$$

for all integers k. Since

$$u_k = U^k u_0 = \int_{\Upsilon} z^k \, dE(z) u_0,$$

our desired vector-valued measure is defined, for A a Borel set, by

$$Z(A) \equiv E(A) u_0.$$

The properties of Z in Definition 2.2 follow from the properties of $E : A \mapsto E(A)x$ is a vector-valued measure, for all x in the domain of U, $E(\phi)$ is the zero operator, $E(\mathbf{C}) = I$, and $E(A)E(B) = E(A \cap B)$.
(c) $\rightarrow$ (d). For A a Borel set, define

$$\mu(A) \equiv \langle Z(A), u_0 \rangle \, .$$

The discussion after Definition 2.2 shows that this is a positive measure. A standard style of measure-theoretic argument shows that

$$\left\langle \int_{\Upsilon} f(z) \, dZ(z), \int_{\Upsilon} g(z) \, dZ(z) \right\rangle = \int_{\Upsilon} f(z) \overline{g(z)} \, d\mu(z), \tag{*}$$

for Borel meaurable functions f, g on Υ, by first showing it for simple functions: if $f = \sum_k \alpha_k 1_{A_k}$ and $g = \sum_j \beta_j 1_{B_j}$,

$$\left\langle \int_{\Upsilon} f(z) \, dZ(z), \int_{\Upsilon} g(z) \, dZ(z) \right\rangle = \sum_{k,j} \alpha_k \overline{\beta_j} \, \langle Z(A_k), Z(B_j) \rangle = \sum_{k,j} \alpha_k \overline{\beta_j} \, \langle Z(A_k \cap B_j), u_0 \rangle$$
$$\equiv \sum_{k,j} \alpha_k \overline{\beta_j} \mu(A_k \cap B_j) = \int_{\Upsilon} f(z) \overline{g(z)} \, d\mu(z),$$

thus (*) holds for f, g simple. For general f, g, write both as the limits of simple functions to get (*).

In particular, since $\overline{z^m} = z^{-m}$ for $z \in \Upsilon$,

$$\langle u_n, u_m \rangle = \left\langle \int_\Upsilon z^n \, dZ(z), \int_\Upsilon z^m \, dZ(z) \right\rangle = \int_\Upsilon z^{n-m} \, dZ(z),$$

for any integers n, m.

(d) $\to$ (a). pretty obvious.

Remark 2.4. In Theorem 2.3, the closure of the span of $\{u_k\}_{k \in \mathbf{Z}}$ is then unitarily equivalent to $L^2(\Upsilon, \mu)$, and U is unitarily equivalent to multiplication by z. This is another form of the spectral theorem, as stated in [Re-S, Theorem VII.3] for self-adjoint operators, but for completeness I will put the straightforward argument here.

Explicitly, define Λ, from the span of $\{u_k\}_{k \in \mathbf{Z}}$ into $L^2(\Upsilon, \mu)$ by

$$\Lambda\left(\sum_{k=m}^{n} \alpha_k u_k\right) \equiv \left(z \mapsto \sum_{k=m}^{n} \alpha_k z^k\right) \quad (\{\alpha_k\}_{k=m}^{n} \subseteq \mathbf{C}),$$

then

$$\|\Lambda\left(\sum_{k=m}^{n} \alpha_k u_k\right)\|^2 = \left\langle \Lambda\left(\sum_{k=m}^{n} \alpha_k u_k\right), \Lambda\left(\sum_{=m}^{n} \alpha_j u_j\right) \right\rangle = \sum_{k,j} \alpha_k \overline{\alpha_j} \langle u_k, u_j \rangle = \sum_{k,j} \alpha_k \overline{\alpha_j} \int_\Upsilon z^{k-j} \, d\mu(z)$$
$$= \int_\Upsilon |\sum_{k=m}^{n} \alpha_k z^k|^2 \, d\mu(z);$$

that is, Λ is an isometry onto the set of polynomials, a dense subspace of $L^2(\Upsilon, \mu)$, hence extends to a unitary operator from the closure of the span of $\{u_k\}_{k \in \mathbf{Z}}$ onto $L^2(\Upsilon, \mu)$.

For the unitary equivalence of U, note that, for any integer $k, z \in \Upsilon$,

$$(\Lambda U u_k)(z) = (\Lambda u_{k+1})(z) = z^{k+1} = z(\Lambda u_k)(z).$$

Example 2.5. A time series is (weakly) stationary (see [B-Da, Definition 1.3.2]) if and only if it is unitary, as in Definition 2.2. (c) and (d) of Theorem 2.3 are precisely the spectral representation of the time series, as constructed in [B-Da, Chapter 4]; Z of Theorem 2.3(c) is the "orthogonal increment process" associated with the time series.

Note that the autogregressive moving average (ARMA) process ([B-Da, Definition 3.1.2])

$$X_k - \phi_1 X_{n-1} - \cdots - \phi_p X_{k-p} = Z_k + \theta_1 Z_{k-1} + \cdots + \theta_p Z_{k-q},$$

where $k \in \mathbf{Z}, p, q \in \mathbf{N}$, and $\{Z_j\}_{j \in \mathbf{Z}}$ is "white noise," that is, orthogonal, with mean zero and constant variance, may be written as

$$P(U^{-1} X_k) = Q(U^{-1} Z_k),$$

where U is from Theorem 2.3, P and Q are polynomials

$$P(z) \equiv 1 - \phi_1 z - \cdots \phi_p z^p, \quad Q(z) \equiv 1 - \theta_1 z - \cdots \theta_q z^q.$$

The ARMA may then be immediately solved as

$$X_k \equiv \frac{P}{Q}(U^{-1}) Z_k,$$

and constructed by the integral representations in Theorem 2.3(c) and (d), with μ a constant times Lebesgue measure on the unit circle Υ, when P has no zeroes on the unit circle. More precisely, the sequence $\{X_k\}_{k \in \mathbf{Z}}$ is unitarily equivalent to the sequence of functions $\{(z \mapsto \frac{P}{Q}(\frac{1}{z}))\}_{k \in \mathbf{Z}}$ in $L^2(\Upsilon, \mu)$.

Theorem 2.6. *Suppose* $\{x_k\}_{k=0}^{\infty}$ *is an orthogonal set and*

$$u_n \equiv \sum_{k=0}^{n} x_k \ \ (n = 0, 1, 2, ...).$$

Then there exists linear U *on the span of* $\{x_k\}_{k=0}^{\infty}$ *such that* $u_n = U^n u_0$, *for* $n \in \mathbf{N}$.

U *is bounded if and only if*

$$\sup_{j \geq 0} \frac{\|x_{j+1}\|}{\|x_j\|} < \infty;$$

we then have

$$\|U\|^2 = \max\{(1 + \frac{\|x_1\|^2}{\|x_0\|^2}), \sup_{j \geq 1} \frac{\|x_{j+1}\|^2}{\|x_j\|^2}\},$$

and U *extends to* $\{\sum_{k=0}^{\infty} \beta_k x_k \mid \sum_{k=0}^{\infty} |\beta_k|^2 \|x_k\|^2 < \infty\}$.

Proof: Suppose, for complex $\{\alpha_j\}_{j=0}^{N}$,

$$0 = \sum_{n=0}^{N} \alpha_n u_n = \sum_{n=0}^{N} \sum_{k=0}^{n} \alpha_n x_k = \sum_{k=0}^{N} \left(\sum_{n=k}^{N} \alpha_n \right) x_k.$$

By orthogonality, $\sum_{n=k}^{N} \alpha_n = 0$, for $0 \leq k \leq N$. This implies that $\alpha_n = 0$, for $0 \leq n \leq N$; that is, $\{u_n\}_{n=0}^{\infty}$ is linearly independent. By Proposition 1.2, there exists linear U, on the span of $\{u_n\}$ such that $u_n = U^n u_0$, for n a nonnegative integer.

Since $x_0 = u_0$ and $x_k = u_k - u_{k-1}$, for $k \in \mathbf{N}$, the span of $\{x_k\}_{k=0}^{\infty}$ equals the span of $\{u_n\}_{n=0}^{\infty}$, with

$$Ux_0 = x_0 + x_1, \ Ux_k = x_{k+1}(k \in \mathbf{N}). \tag{*}$$

For $\beta_k, k = 0, 1, 2, ...$, complex, $\sum_{k=0}^{\infty} |\beta_k|^2$ finite, denote

$$x = \sum_{k=0}^{\infty} \beta_k x_k.$$

Let

$$K \equiv \max\{(1 + \frac{\|x_1\|^2}{\|x_0\|^2}), \sup_{j \geq 1} \frac{\|x_{j+1}\|^2}{\|x_j\|^2}\}.$$

By orthogonality and (*),

$$\|x\|^2 = \sum_{k=0}^{\infty} |\beta_k|^2 \|x_k\|^2$$

and

$$\|Ux\|^2 = |\beta_0|^2(\|x_0\|^2 + \|x_1|^2) + \sum_{k=1}^{\infty} |\beta_k|^2 \|x_{k+1}\|^2 = (1 + \frac{\|x_1\|^2}{\|x_0\|^2})|\beta_0|^2 \|x_0\|^2 + \sum_{k=1}^{\infty} (\frac{\|x_{k+1}\|^2}{\|x_k\|^2}\})|\beta_k|^2 \|x_k\|^2 \leq K\|x\|^2$$

Thus U is bounded when $\sup_{j \in \mathbf{N}} \frac{\|x_{j+1}\|}{\|x_j\|}$ is finite, and $\|U\|^2 \leq K$. Since

$$\|Ux_0\|^2 = \|x_0\|^2 + \|x_1\|^2 = (1 + \frac{\|x_1\|^2}{\|x_0\|^2})\|x_0\|^2,$$

and, for $j \in \mathbf{N}$,

$$\|Ux_j\|^2 = \|x_{j+1}\|^2 = \left(\frac{\|x_{j+1}\|^2}{\|x_j\|^2} \right) \|x_j\|^2,$$

$\|U\|^2 \geq K$, so that $\|U\|^2 = K$, as desired. $\square$

Example 2.7. A random walk is a stochastic process $\{u_n\}_{n=0}^{\infty}$, where $u_n = \sum_{k=0}^{n} x_k$, for $\{x_k\}_{k=0}^{\infty}$ independent, hence uncorrelated, that is, orthogonal, random variables. In fact, the x_ks are commonly identically distributed, so that $\|x_j\| = \|x_{j+1}\|$, for all j, thus, for U as in Theorem 2.6, $\|U\|^2 = 2$.

Remark 2.8. One could also, in Theorem 2.6, use Proposition 1.2 to construct a linear operator T such that $Tx_k = x_{k+1}$, for $k = 0, 1, 2, ...$ Then $U = P + T$, where P is the one-dimensional projection onto the span of $\{x_0\}$ (see (*) in the proof of Theorem 2.6).

REFERENCES

[B-Da] P. J. Brockwell and R. A. Davis, "Time Series: Theory and Methods," Springer, New York, 1991.

[Di-U] J. Diestel and J.J. Uhl, Jr., "Vector Measures," Math. Surveys and Monographs, No. 15, Amer. Math. Soc., Providence, Rhode Island, 1977.

[Re-S] M. Reed and B. Simon, "Methods of Modern Mathematical Physics, Volume I," Academic Press, New York, 1972.

[Ru] W. Rudin, "Functional Analysis," McGraw-Hill, 1973.

Ohio State University, Department of Statistics, Columbus, Ohio 43210 USA

E-mail: delaubenfels.1@osu.edu