Algebra of growth orders

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April 7, 2024

Abstract

In this thesis we offer a nontraditional approach to asymptotic analysis and the theory of growth orders that is uniquely algebraic in spirit and avoids relying on machinery from calculus. Rather than focusing just on specific applications and computations, we treat growth orders as algebraic objects in their own right, giving rise to both a plethora of formulas in the traditional "cookbook" style, and a deeper understanding of the structure of the space of growth orders. This gives us a novel angle from which to approach questions about, for instance, comparability of growth orders and asymptotic estimates of partial sums or subsequences of sequences.

Contents

1. Introduction

The author's naive impression is that, in most of the available literature, when asymptotic analysis makes an appearance, it is most often meant to shed light on some *other* object of interest, such as an algorithm, a special function, or a number-theoretic sequence. Rarely are asymptotic growth orders themselves the center of attention. [1] provides one of the few books that treats asymptotic growth orders exclusively, and even then, it is very application-oriented, containing more heuristics and worked-out examples than broad theoretical results.

As [8] mentions, one of our goals is to produce a family of "cookbook-like" formulas in the style of [4] that yield asymptotic estimates for large families of sequences defined, say, by sums or recurrence relations. However, our approach will differ philosophically in that, rather than treating a growth order as a property of a sequence, we will often treat growth orders as algebraic objects of their own right. Although we will often have to manipulate specific sequences in order to construct proofs or counterexamples, we can sometimes avoid "getting in the weeds" with messy bounding arguments and prove propositions in a more elegant way.

We will often study functions/properties of sequences that depend only on their growth order, in order to extend their definitions to well-defined operations on growth orders themselves. For instance, we show that the growth orders of $(a_n + b_n)$, $(a_n \cdot b_n)$, and $(\Sigma_{k=1}^{n} a_k)$ depend only on the growth orders of (a_n) and (b_n), and find necessary and sufficient conditions on (a_n) to guarantee that the growth order of (a_{b_n}) depends only on that of (b_n). This allows us to extend the operations of addition, multiplication, partial summation, and function composition on sequences to growth orders in a well-defined way.

Finally, in addition to generating a "cookbook" of formulas, we will also be interested in getting a birds'-eye-view of the algebraic structure of growth-order relations and operations. For instance, after defining a partial ordering on growth orders, we may ask questions about certain suborders or cuts of this partial ordering. After defining addition on growth orders, we will see that it constitutes the join operation for a lattice structure. The growth orders $> \mathbf{1}$ comprise a monoid under multiplication, and another special class of growth orders comprises a group under composition, with composition distributing over multiplication, and interacting with addition to form a lattice-ordered group, revealing even richer algebraic structure. All of these observations give rise to questions such as the following:

- Is there a minimal growth order for sequences with divergent partial sums? (No.)
- What is the smallest ordinal that is not the order type of some chain of the poset of growth orders? (Unknown.)
- Is the lattice of growth orders a complete lattice? (No.)

- Is the group of composable growth orders under composition a torsion-free group? (Yes.)

In addition to identifying the properties of these naturally-occurring algebraic structures, we shall also try to manufacture structures with certain "desirable properties". Some of these properties include

- "Well-behavedness" conditions of the constituent growth orders
- Closure under operations such as sums, products, composition, or taking partial sums
- Containing solutions to certain growth-order equations
- Trichotomy, i.e. having pairwise comparable growth orders

Another philosophical difference between this thesis and other traditional approaches to asymptotic analysis is that we avoid the preexisting machinery of calculus (e.g. derivatives and integrals) whenever possible. This makes our treatment accessible to any student who is comfortable with real arithmetic and inequalities, basic set and order theory (in particular the use of equivalence relations), and indirect proof techniques. Although derivatives and integrals are powerful tools in asymptotic analysis, as the saying goes, *when you have a hammer, everything looks like a nail* - having access to these tools may obscure the intrinsic structure and beauty of growth orders and preclude a search for alternatives, so we prefer to avoid them in this exploration when possible.

Finally, here is a sketch of the progression of topics in this thesis, by chapter:

- **Defining growth orders.** We rigorously define what exactly we mean by "a growth order" and establish notational conventions, and well as introduce a "niceness condition" called *moderateness* that will play a key role in the rest of the thesis, particularly the chapter on composition.
- **Partial ordering.** We define how growth orders are to be compared, and make note of the important fact that not all pairs of growth orders are comparable at all.
- **Arithmetic.** We explain how to extend familiar operations on sequences to operations on growth orders, in the typical way that functions are defined on equivalence classes, and show that this cannot be done naively by providing examples of operations for which this breaks down. We also take a closer look at the partial ordering on growth orders and how it interacts with these new operations.
- **Partial summation.** We define the first operation on growth orders that is not analogous to an operation on the real numbers, namely partial summation. We also define another important "niceness condition" called *monotonicity* and show how both monotonicity and moderateness can simplify the computation of certain growth orders.
- **Composition and inverses.** We determine necessary and sufficient conditions under which the growth order of the *composite* of two sequences depends uniquely on the sequences' growth orders, then explore how this new binary operation interacts with those previously defined. We uncover a beautiful lattice-ordered group structure on a special set of growth orders.

- **Closed chains.** Rather than defining new operations, we tackle the issue of trying to construct "good environments" for doing algebra with growth orders. Namely, we seek sets of growth orders that are "sufficiently large" to be closed under certain desirable operations like multiplication or partial summation, while being "sufficiently small" to be totally ordered. We also establish several "cookbook" formulas for partial summation. The algorithmic results from this chapter were used to implement an asymptotic growth order calculator that includes several operations on growth orders discussed in this thesis [3].

2. Defining growth orders

2.1. Definition of a growth order

We will use $\mathcal{S}(\mathbb{R}^+)$ to denote the set of sequences of positive real numbers. Notice that this is not what would usually be called a sequence space, because it is *not even a vector space*. It comes with operations of addition and scalar multiplication, but the underlying set of scalars $\mathbb{R}^+$ is not a field, as it lacks a zero element and additive inverses. We have sacrificed these properties for a reason: growth orders are meant to formalize the notion of asymptotic relative growth of sequences, and the negative real numbers and 0 are not amenable to the concept of "relative size".

Definition 1. Let $\alpha = (a_n)$ and $\beta = (b_n)$ be two sequences in $\mathcal{S}(\mathbb{R}^+)$. We will say that α, β **have the same growth order**, or $\alpha \sim \beta$, if there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 b_n \leq a_n \leq C_2 b_n$$

for all $n \in \mathbb{N}$.

There are several equivalent ways of thinking of this definition. The statement $\alpha \sim \beta$ is easily shown to be equivalent to each of the following:

- Both of the quotients a_n/b_n and b_n/a_n are bounded above in $\mathbb{R}^+$.
- The quotient a_n/b_n is bounded above and below by two strictly positive constants.
- $a_n = \Theta(b_n)$, or equivalently $b_n = \Theta(a_n)$, for those who are familiar with asymptotic notation.
- Both $a_n = O(b_n)$ and $b_n = O(a_n)$. [1]
- The following limits are finite and positive:

$$\limsup_{n\to\infty} \frac{a_n}{b_n} \qquad \liminf_{n\to\infty} \frac{a_n}{b_n}$$

Now we shall prove that $\sim$ defines an *equivalence relation* on $\mathcal{S}(\mathbb{R}^+)$. The equivalence classes, consisting of all sequences with the same growth order, will be the objects that we refer to as *growth orders*.

[1] $a_n = O(b_n)$ gives us the existence of the constant C_2 for the upper bound on a_n, and $b_n = O(a_n)$ gives us the existence of the constant C_1 for the lower bound on a_n.

Proposition 2. The relation $\sim$ on sequences in $\mathcal{S}(\mathbb{R}^+)$ is an equivalence relation.

Proof. Reflexivity follows directly from the definition. Setting $C_1 = C_2 = 1$ immediately gives us $C_1 a_n \leq a_n \leq C_2 a_n$ for all $n \in \mathbb{N}$, and therefore $(a_n) \sim (a_n)$ for all sequences $(a_n) \in \mathcal{S}(\mathbb{R}^+)$.

To show symmetricity, suppose that $(a_n) \sim (b_n)$ for some sequences $(a_n), (b_n) \in \mathcal{S}(\mathbb{R}^+)$. Then we have that

$$C_1 b_n \leq a_n \leq C_2 b_n$$

for some $C_1, C_2 \in \mathbb{R}^+$. This implies that

$$C_2^{-1} a_n \leq b_n \leq C_1^{-1} a_n$$

and therefore $(b_n) \sim (a_n)$, as desired.

Finally, to show transitivity, suppose that $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$ for some sequences $(a_n), (b_n), (c_n) \in \mathcal{S}(\mathbb{R}^+)$. Then we have that

$$C_1 b_n \leq a_n \leq C_2 b_n$$

$$C_3 c_n \leq b_n \leq C_4 c_n$$

for some constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$. This implies that

$$C_1 C_3 c_n \leq a_n \leq C_2 C_4 c_n$$

and therefore $(a_n) \sim (c_n)$, so that transitivity holds as claimed. □

Now we are prepared to define growth orders as *equivalence classes*:

Definition 3. A **growth order** is defined as an equivalence class belonging to $\mathcal{S}(\mathbb{R}^+)/\sim$. If $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$, then the **growth order of** α is the equivalence class of α under $\sim$, and may be denoted $[\alpha]$ or $[a_n]$.

We will often use Latin letters like a_n to refer to elements of sequences, use Greek letters like α to refer to sequences, and use Fraktur letters like $\mathfrak{a}$ to refer to equivalence classes of sequences, i.e. growth orders.

Before moving on, we make a brief observation: that changing only finitely many terms of a sequence does not affect its growth order.

Proposition 4. If $(a_n), (a_n') \in \mathcal{S}(\mathbb{R}^+)$ differ in only finitely many positions, then $(a_n) \sim (a_n')$, so that they have the same growth order.

Proof. Suppose that $(a_n), (a'_n) \in \mathcal{S}(\mathbb{R}^+)$ differ in only finitely many positions. Then there exists $N \in \mathbb{N}$ such that $a_n = a'_n$ for all $n > N$. We may therefore define constants $C_1, C_2 > 0$ as follows:

$$C_1 = \max_{0 \le i \le N} \frac{a'_i}{a_i}$$
$$C_2 = \min_{0 \le i \le N} \frac{a'_i}{a_i}$$

Then it follows that $a'_n \le C_1 a_n$ for all $n \le N$, and $a'_n \ge C_2 a_n$ for all $n \le N$. Further, since $a'_n = a_n$ for all $n > N$, we have that $a'_n \le \max(C_1, 1) a_n$ for all $n \in \mathbb{N}$, and similarly $a'_n \ge \min(C_2, 1) a_n$ for all $n \in \mathbb{N}$. This proves that $(a_n) \sim (a'_n)$ by definition. □

This means that when defining a growth order as the equivalence class of a specific sequence, it suffices to specify the values of that sequence for *all but finitely many* entries, since the growth order is invariant under changes of finitely many entries. For instance, we may refer to "the growth order of the sequence $(\sqrt{n-10})$" even though the expression $\sqrt{n-10}$ does not evaluate to a positive real number when $n \le 10$. As a matter of convention, when we write something like this, we are referring to the common growth order of all sequences whose entries are given by the provided expression when that expression is defined, positive and real.

2.2. Common growth orders

This is just a short section meant to establish notation that we will use later in the write-up to denote some commonly-occurring growth orders.

- $\mathbf{1}$ denotes the constant growth order $[(1)]$.
- $\mathfrak{n}$ denotes the growth order $[(n)]$.
- $\mathfrak{n}^p$ denotes the growth order $[(n^p)]$, for $p \in \mathbb{R}^+$.
- $\mathfrak{u}$ denotes the pathological growth order $[(n^{(-1)^n})]$, which we will often use as a counterexample because of its oscillatory nature.
- $\mathfrak{l}$ denotes the growth order $[(\log n)]$.
- $\mathfrak{l}_m$ denotes the growth order $[(\log \cdots \log n)]$, where there are m nested logs.
- $\mathfrak{l}(p_0, p_1, \cdots, p_m)$ denotes the growth order of the sequence

$$n^{p_0} (\log n)^{p_1} \cdots (\overbrace{\log \cdots \log}^{m \text{ nested logs}} n)^{p_m}$$

Notice that the expressions $\log n$, $\log \log n$ and so on are not generally positive real numbers for all $n \in \mathbb{N}$. Our comment at the end of the previous section explains why we may still use them to define growth orders.

2.3. Moderate growth orders

Now that we have defined growth orders in general, we will define a certain subclass of sequences that we will spend extra time exploring because of their favorable properties.

Definition 5. Let us call $\alpha \in \mathcal{S}(\mathbb{R}^+)$ a sequence of **moderate growth** if, for any $k \in \mathbb{N}$, there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m \leq kn$. (Note that C_1, C_2 may depend on k.)

Notice the similarity between our moderateness condition and the "regularity condition" of [8], where it is used to solve certain divide-and-conquer recurrences. For us, this "niceness" condition will become useful, for instance, in Section 5.1, where we define the composition of two sequences.

Although it is not immediately obvious from the definition, a necessary condition for moderate growth is polynomial or sub-polynomial growth/decay. To be precise, every moderate sequence is bounded between power-sequences of the form (Cn^p) with $p \in \mathbb{R}$ and $C \in \mathbb{R}^+$.

Proposition 6. If a sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$ exhibits moderate growth, then there exist $p, q \in \mathbb{R}$ and $C_1', C_2' \in \mathbb{R}^+$ such that

$$C_1' n^p \leq a_n \leq C_2' n^q$$

for all $n \in \mathbb{N}$. The converse is not true.

Proof. Suppose that (a_n) exhibits moderate growth. Then let C_1, C_2 be constants such that

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n \leq m \leq 2n$. It is automatic that $C_1 \leq 1$ and $C_2 \geq 1$, by considering the case of $m = n$. Inductively, we may show that for any $n \in \mathbb{N}$,

$$a_n \leq C_2 a_{\lceil n/2 \rceil} \leq C_2^2 a_{\lceil n/4 \rceil} \leq \cdots \leq C_2^{\lceil \log_2 n \rceil} a_1 \leq a_1 C_2 \cdot n^{\log_2 C_2}$$

where we have used the fact that $\lceil \log n \rceil \leq \log n + 1$ and the identity $x^{\log y} = y^{\log x}$ for positive reals x, y in the last step of the above chain of inequalities. Similarly

$$a_n \geq C_1 a_{\lceil n/2 \rceil} \geq C_1^2 a_{\lceil n/4 \rceil} \geq \cdots \geq C_1^{\lceil \log_2 n \rceil} a_1 \geq a_1 \cdot n^{\log_2 C_1}$$

where we have used the fact that $\lceil \log n \rceil \geq \log n$. So we have

$$a_1 \cdot n^{\log_2 C_1} \leq a_n \leq a_1 C_2 \cdot n^{\log_2 C_2}$$

which proves the first claim, by taking $p = \log_2 C_1$ and $q = \log_2 C_2$, and $C_1' = a_1$ and $C_2' = a_1 C_2$.

To see why the converse is not true, consider, for instance, the sequence $a_n = n^{(-1)^n}$. For all odd $m \in \mathbb{N}$, we have $a_m = 1/m$, whereas $a_m = m$ for all even m, meaning that for odd m, the quantity $a_{2m}/a_m = 2m^2$ is unbounded, and (a_n) does not satisfy the moderate growth property. □

Here are some propositions that provide sufficient (equivalent) conditions for moderacy that have less stringent requirements, and are therefore easier to prove for some sequences.

Proposition 7. In order for $\alpha \in \mathcal{S}(\mathbb{R}^+)$ to exhibit moderate growth, it is sufficient for there to exist $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m \leq 2n$. In other words, it suffices to find such constants for the case of $k = 2$ in the definition of moderate growth.

Proof. Suppose that such constants $C_1, C_2 > 0$ exist for $k = 2$, and suppose WLOG that $C_1 < 1$ and $C_2 > 1$ (for if not, we may decrease C_1 below 1 and increase C_2 above 1, weakening the inequality). Then we may show by induction that for all $q \in \mathbb{N}$, and for all $m, n \in \mathbb{N}$ with $n \leq m \leq 2^q n$, the following inequality holds:

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

We will prove this by induction on q. Notice that the base case of $q = 1$ is precisely our hypothesis, so we may skip to the inductive step.

As our inductive hypothesis, we assume that this holds for some value of q. By hypothesis, we also know that

$$C_1 a_{2^q n} \leq a_m \leq C_2 a_{2^q n}$$

for all $2^q n \leq m \leq 2^{q+1} n$ which is a direct consequence of our original assumption, in which n is replaced by $2^q n$. But since $C_1^q a_n \leq a_{2^q n} \leq C_2^q a_n$ by the inductive hypothesis, we have that

$$C_1^{q+1} a_n \leq C_1 a_{2^q n} \leq a_m \leq C_2 a_{2^q n} \leq C_2^{q+1} a_n$$

and thus

$$C_1^{q+1} a_n \leq a_m \leq C_2^{q+1} a_n$$

for all $2^q n \leq m \leq 2^{q+1} n$. Since the tighter inequality

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

holds for $n \leq m \leq 2^q n$ by the inductive hypothesis, we may combine the two cases of $n \leq m \leq 2^q n$ and $2^q n \leq m \leq 2^{q+1} n$ and state that

$$C_1^{q+1} a_n \leq a_m \leq C_2^{q+1} a_n$$

for all $n \leq m \leq 2^{q+1} n$.

Thus, the truth of our inequality for some $q \in \mathbb{N}$ implies its truth for $q + 1$. But the base case of $q = 1$ was taken as an assumption, so we have by induction that for all $q \in \mathbb{N}$ and $n \leq m \leq 2^q n$ the inequality

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

holds. Since, for all $k \in \mathbb{N}$, there exists $q \in \mathbb{N}$ such that $2^q \geq k$, if some $k \in \mathbb{N}$ is given, we may choose such a value of $q \in \mathbb{N}$, and then bound

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

for all $n \leq m \leq kn \leq 2^q n$, demonstrating that the sequence (a_n) is moderate by definition. □

Proposition 8. (This is a further weakening of Proposition 7.) In order for $\alpha \in \mathcal{S}(\mathbb{R}^+)$ to exhibit moderate growth, it is sufficient for there to exist $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a_m \leq C_2 a_n$$

for all $n, m \in \mathbb{N}$ with $n \leq m \leq \lceil rn \rceil$ for some $r > 1$.

Proof. Using a similar argument as shown in the above proof, we may show that if this is true for some $r > 1$ with $C_1 < 1$ and $C_2 > 1$, then it follows that

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

for all $n \leq m \leq \lceil r^q n \rceil$, again using a proof by induction. (We must also use the fact that $\lceil s \lceil tn \rceil \rceil \geq \lceil stn \rceil$ for $s, t > 1$ and $n \in \mathbb{N}$.) Because $r > 1$, there exists $q \in \mathbb{N}$ such that $r^q \geq 2$, so that we have

$$C_1^q a_n \leq a_m \leq C_2^q a_n$$

for all $n \leq m \leq 2n \leq \lceil r^q n \rceil$. The result then follows from Proposition 7. □

We may also refer to *growth orders* as being moderate, depending on whether or not they consist of sequences of moderate growth. We will now prove that moderateness is a bonafide property of growth orders by showing that the moderateness of a sequence is completely determined by its growth order.

Proposition 9. Let $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ with $[\alpha] = [\alpha']$. Then α exhibits moderate growth iff α' exhibits moderate growth.

Proof. Suppose $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ with $[\alpha] = [\alpha']$, so that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

for all $n \in \mathbb{N}$ for some $C_1, C_2 \in \mathbb{R}^+$. Suppose that α exhibits moderate growth, so that for each $k \in \mathbb{N}$, there exist constants $C_3, C_4 \in \mathbb{R}^+$ such that

$$C_3 a_n \leq a_m \leq C_4 a_n$$

for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$ between n and kn. Then, if $k \in \mathbb{N}$ is fixed, and $n \leq m \leq kn$ for some $m, n \in \mathbb{N}$, we have

$$a'_m \leq C_2 a_m \leq C_2 C_4 a_n \leq \frac{C_2 C_4}{C_1} a'_n$$

and

$$a'_m \geq C_1 a_m \geq C_1 C_3 a_n \geq \frac{C_1 C_3}{C_2} a'_n$$

so we have that

$$\frac{C_1 C_3}{C_2} a'_n \leq a'_m \leq \frac{C_2 C_4}{C_1} a'_n$$

and therefore α' has moderate growth. Thus, moderate growth of α implies moderate growth of α' and vice versa (by symmetry). □

The following definition is therefore justified:

Definition 10. A growth order $\mathfrak{a}$ is said to be **moderate** if each of its sequences has moderate growth, and **not moderate** (or **immoderate**) if none of its sequences has moderate growth.

Moderate growth sequences have some convenient properties that we will come to appreciate more when it is time to define the composition operation in Section 5.1. For now, however, we can state and prove a few of their elementary properties.

Proposition 11. If $\alpha = (a_n)$ is a moderate growth sequence, then every arithmetic subsequence (a_{jn+k}) with $j, k \in \mathbb{N}$ has the same growth order.

Proof. If $j, k, n \in \mathbb{N}$, then we have $n \leq jn + k \leq (j + k)n$, so we have $(a_n) \sim (a_{jn+k})$ by the moderate growth property of α. □

What sorts of horrible sequences *do not* have this property, you might ask? One example is the pathological sequence $a_n = n^{(-1)^n}$ that was used as a counterexample earlier. However, there are also many naturally-occurring sequences without this property, such as rapidly-growing sequences like $a_n = 2^n$, for which $(a_{2n}) > (a_n)$.

There are, of course, growth orders that are not translation-invariant. The classic pathological example $a_n = n^{(-1)^n}$ works here as well, but another example that feels less contrived is the sequence $a_n = 2^{n^2}$.

We will not consider partial summation in great detail until Section 4.1, but the following proposition serves as one demonstration of the utility of having a repertoire of growth orders that are known to be moderate.

Proposition 12. If $\alpha = (a_n) \in \mathfrak{a}$ is moderate, then

$$\left(\sum_{i=n}^{kn} a_i\right) \sim (na_n)$$

for any $k \in \mathbb{N}$.

Proof. Given $k \in \mathbb{N}$, if (a_n) is moderate, then we have constants C_1, C_2 such that $C_1 a_n \leq a_m \leq C_2 a_n$ for all m with $n \leq m \leq kn$. Thus, we have that

$$\sum_{i=n}^{kn} a_i \leq C_2 \sum_{i=n}^{kn} a_n = C_2(kn - n + 1)a_n$$

and this upper bound is, of course, $\sim (na_n)$. On the other hand, we also have that

$$\sum_{i=n}^{kn} a_i \geq C_1 \sum_{i=n}^{kn} a_n = C_1(kn - n + 1)a_n$$

so we also have a lower bound that is $\sim (na_n)$. Hence, we have that

$$\left(\sum_{i=n}^{kn} a_i\right) \sim (na_n)$$

as claimed. ◻

Knowing only that a sequence is moderate gives us an easy shortcut for evaluating sums of the above form - just multiply them by $\mathfrak{n}$. This allows us to immediately deduce asymptotic formulas such as the following:

$$\sum_{k=n}^{2n} \frac{\log^2 k}{k} = \Theta(\log^2 n)$$

...provided, of course, that $(\log^2 n/n)$ is a moderate sequence. This, however, is not difficult to show. By Proposition 7, it suffices to show that

$$C_1 \frac{\log^2 n}{n} \leq \frac{\log^2 m}{m} \leq C_2 \frac{\log^2 n}{n}$$

for some $C_1, C_2 > 0$, for all $n \leq m \leq 2n$ and n sufficiently large. Since the logarithm is monotone increasing, we have that

$$\log^2(m) \leq \log^2(2n) \leq \log^2(n^2) = 4\log^2(n)$$

for all $n \leq m \leq 2n$ and $n \geq 2$. This means that

$$\frac{1}{2} \cdot \frac{\log^2 n}{n} \leq \frac{\log^2 m}{m} \leq 4 \cdot \frac{\log^2 n}{n}$$

completing the proof of moderateness, from which the claimed summation formula follows.

3. Arithmetic

3.1. Sums, products and quotients

In this section, we will prove that the elementwise arithmetic operations of $+, \cdot, \div$ on sequences in $\mathcal{S}(\mathbb{R}^+)$ can be extended to growth orders in a natural way without accidentally introducing any ill-defined expressions. Let us first define these operations on sequences, and then extend the definition to growth orders:

Definition 13. Given $\alpha = (a_n), \beta = (b_n) \in \mathcal{S}(\mathbb{R}^+)$, define their **elementwise sum** $\alpha + \beta = (a_n + b_n)$, their **elementwise product** $\alpha \cdot \beta = (a_n b_n)$, and their **elementwise quotient** $\alpha/\beta = (a_n/b_n)$. The **elementwise reciprocal** of α is defined as $\alpha^{-1} = (a_n^{-1})$.

Proposition 14. If $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$, then $[\alpha + \beta] = [\alpha' + \beta']$.

Proof. Let $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$. Then there exist constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

$$C_3 b_n \leq b'_n \leq C_4 b_n$$

for all $n \in \mathbb{N}$. By adding these inequalities, we have that

$$C_1 a_n + C_3 b_n \leq a'_n + b'_n \leq C_2 a_n + C_4 b_n$$

and, since a_n, b_n are positive reals, we have

$$\min(C_1, C_3)(a_n + b_n) \leq a'_n + b'_n \leq \max(C_2, C_4)(a_n + b_n)$$

and therefore $\alpha + \beta \sim \alpha' + \beta'$, proving the claim. $\square$

Proposition 15. If $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$, then $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$.

Proof. Let $\alpha, \alpha' \in \mathfrak{a}$ and $\beta, \beta' \in \mathfrak{b}$. Then there exist constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

$$C_3 b_n \leq b'_n \leq C_4 b_n$$

for all $n \in \mathbb{N}$. Multiplying these inequalities yields

$$C_1 C_3 a_n b_n \le a'_n b'_n \le C_2 C_4 a_n b_n$$

so that we immediately have $\alpha \cdot \beta \sim \alpha' \cdot \beta'$. □

Proposition 16. If $\alpha, \alpha' \in \mathfrak{a}$ and $p \in \mathbb{R}$, we have $[\alpha^p] = [\alpha'^p]$.

Proof. Let $\alpha, \alpha' \in \mathfrak{a}$ so that there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 a_n \le a'_n \le C_2 a_n$$

for all $n \in \mathbb{N}$. If $p \in \mathbb{R}$ is nonnegative, then we have ${C_1}^p {a_n}^p \le {a'_n}^p \le {C_2}^p {a_n}^p$ since $x \mapsto x^p$ is a monotone increasing function on $\mathbb{R}^+$, so that $[\alpha^p] = [\alpha'^p]$ automatically. Otherwise, if p is negative we may use the fact that $x \mapsto x^p$ is a decreasing function on $\mathbb{R}^+$, so that ${C_2}^p {a_n}^p \le {a'_n}^p \le {C_1}^p {a_n}^p$ for all $n \in \mathbb{N}$, and $[\alpha^p] = [\alpha'^p]$ in this case as well. □

This means that the growth orders given by $[\alpha+\beta]$, $[\alpha \cdot \beta]$, and $[\alpha^p]$ depend only on the growth orders of α and β, so they may as well be defined as functions of $\mathfrak{a}$ and $\mathfrak{b}$. This leads to the next definition:

Definition 17. Given growth orders $\mathfrak{a} = [\alpha]$ and $\mathfrak{b} = [\beta]$, define their **sum** $\mathfrak{a}+\mathfrak{b} = [\alpha+\beta]$, their **product** $\mathfrak{a} \cdot \mathfrak{b} = [\alpha \cdot \beta]$, and their **quotient** $\mathfrak{a}/\mathfrak{b} = [\alpha/\beta] = [\alpha \cdot \beta^{-1}]$. Define the **reciprocal** of the growth order $\mathfrak{a}$ as $\mathfrak{a}^{-1} = [\alpha^{-1}] = [1/\alpha]$. Given $p \in \mathbb{R}$, define the **power** of $\mathfrak{a}$ raised to the p, or $\mathfrak{a}^p$, as the growth order $[\alpha^p]$.

From the definitions of elementwise addition, products, and quotients, the following familiar algebraic identities immediately follow:

Proposition 18. For all growth orders $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ we have the following identities:

$$\begin{aligned}
\mathfrak{a} + \mathfrak{b} &= \mathfrak{b} + \mathfrak{a} \\
\mathfrak{a} \cdot \mathfrak{b} &= \mathfrak{b} \cdot \mathfrak{a} \\
(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} &= \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) \\
(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c} &= \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c}) \\
\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{c}) &= \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{c} \\
\mathfrak{a} \cdot \mathbf{1} &= \mathfrak{a} \\
\mathfrak{a} \cdot \mathfrak{a}^{-1} &= \mathbf{1} \\
\mathfrak{a} \cdot \mathfrak{b}^{-1} &= \mathfrak{a}/\mathfrak{b}
\end{aligned}$$

Proof. The identities

$$
\begin{aligned}
a + b &= b + a \\
a \cdot b &= b \cdot a \\
(a + b) + c &= a + (b + c) \\
(a \cdot b) \cdot c &= a \cdot (b \cdot c) \\
a \cdot (b + c) &= a \cdot b + a \cdot c \\
a \cdot 1 &= a \\
a \cdot a^{-1} &= 1 \\
a \cdot b^{-1} &= a/b
\end{aligned}
$$

are known to hold for real numbers $a, b, c > 0$. Since sums, products and reciprocals of sequences are computed elementwise, we have the following corresponding identities for any two sequences of positive real numbers α, β, γ:

$$
\begin{aligned}
\alpha + \beta &= \beta + \alpha \\
\alpha \cdot \beta &= \beta \cdot \alpha \\
(\alpha + \beta) + \gamma &= \alpha + (\beta + \gamma) \\
(\alpha \cdot \beta) \cdot \gamma &= \alpha \cdot (\beta \cdot \gamma) \\
\alpha \cdot (\beta + \gamma) &= \alpha \cdot \beta + \alpha \cdot \gamma \\
\alpha \cdot (1) &= \alpha \\
\alpha \cdot \alpha^{-1} &= (1) \\
\alpha \cdot \beta^{-1} &= \alpha/\beta
\end{aligned}
$$

Finally, sums, products and reciprocals of growth orders are defined by corresponding operations on their constituent sequences (which were already proven to be well-defined), the claimed identities follow. ◻

3.2. Preserving moderate growth

In this section we will show briefly that these operations preserve the moderate growth property, so that we may freely take sums and products of moderate growth sequences without worrying about inadvertently producing immoderate growth sequences.

Proposition 19. If $\mathfrak{a}, \mathfrak{b}$ are moderate growth sequences, then $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cdot \mathfrak{b}$ and $\mathfrak{a}^{-1}$ are moderate growth sequences.

Proof. Let $\mathfrak{a}, \mathfrak{b}$ be moderate growth sequences, so that for all $k, m, n \in \mathbb{N}$ with $n \leq m \leq kn$, we have constants $C_1, C_2, C_3, C_4 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a_m \leq C_2 a_n$$

$$C_3 b_n \leq b_m \leq C_4 b_n$$

Then we have

$$\min(C_1, C_3)(a_n + b_n) \leq a_m + b_m \leq \max(C_2, C_4)(a_n + b_n)$$

so that $\mathfrak{a} + \mathfrak{b}$ has moderate growth. We also have

$$C_1 C_3 (a_n \cdot b_n) \leq a_m \cdot b_m \leq C_2 C_4 (a_n \cdot b_n)$$

so that $\mathfrak{a} \cdot \mathfrak{b}$ has moderate growth, and

$$C_2^{-1} a_n^{-1} \leq a_m^{-1} \leq C_1^{-1} a_n$$

so that $\mathfrak{a}^{-1}$ has moderate growth as well. □

3.3. Subtraction and exponentiation

All of the trouble we have gone to in the above sections to define the simple operations of addition, multiplication, and division might seem overly pedantic. After all, these operations extend to growth orders exactly how we'd expect them to, and their properties are more or less what we would expect. So why go to all this trouble to show that they are well-defined? As it happens, not all operations from real arithmetic extend so nicely to $\mathcal{S}(\mathbb{R}^+)$, and in this section we will briefly discuss two examples: subtraction and exponentiation.

After defining addition on growth orders, it seems a natural next step to attempt a definition of subtraction. Perhaps we could define $\mathfrak{a} - \mathfrak{b}$ as the growth order of the sequence $(a_n - b_n)$. An obvious issue is that the difference $a_n - b_n$ may be negative or zero, and therefore $\notin \mathbb{R}^+$. This could be remedied by considering instead the absolute difference $|a_n - b_n|$, but we shall see that this approach is not viable either.

Consider the following three sequences:

$$\begin{aligned} a_n &= n + \frac{1}{n} + \frac{1}{n^2} \\ b_n &= n \\ b'_n &= n + \frac{1}{n} \end{aligned}$$

Then we have that $(b_n) \sim (b'_n)$, while $(a_n - b_n) \sim (1/n)$ and $(a_n - b'_n) \sim (1/n^2)$, which *do not* have the same growth order. That illustrates why the growth order of the difference $(a_n - b_n)$ *does not* depend only on the growth orders of (a_n) and (b_n), and therefore the difference $\mathfrak{a} - \mathfrak{b}$ cannot be well-defined.

In fact, $\mathcal{S}(\mathbb{R}^+)/\sim$ does not have a *cancellation law*, so it is impossible in principle to define an operation $-$ satisfying $(\mathfrak{a} + \mathfrak{b}) - \mathfrak{b} = \mathfrak{a}$ - which is something that we would certainly want subtraction to satisfy if we were to define it! We will see in Section 3.4 when we define a partial ordering on growth orders that if $\mathfrak{b}_1, \mathfrak{b}_2$ are two distinct growth orders both $\leq \mathfrak{a}$, then we have $\mathfrak{a} + \mathfrak{b}_1 = \mathfrak{a} + \mathfrak{b}_2 = \mathfrak{a}$ so that if a cancellation law were to exist, then we would have

$$\mathfrak{b}_1 = (\mathfrak{a} + \mathfrak{b}_1) - \mathfrak{a} = \mathfrak{a} - \mathfrak{a} = (\mathfrak{a} + \mathfrak{b}_2) - \mathfrak{a} = \mathfrak{b}_2$$

which is a contradiction! (The interesting question of whether subtraction could be reasonably extended to $\mathcal{S}(\mathbb{R}^+)/\sim$ was posed by Nic Berkopec.)

Exponentiation is another example: we cannot define $\mathfrak{a}^{\mathfrak{b}}$ as the growth order of $(a_n^{b_n})$, because this is not uniquely defined by the growth orders of (a_n) and (b_n). For instance, consider $a_n = 2$, $a'_n = 3$, $b_n = n$, and $b'_n = 2n$. Then the sequences $({a_n}^{b_n})$, $({a'_n}^{b_n})$, $({a_n}^{b'_n})$, and ${a'_n}^{b'_n}$ all have different growth orders, causing the desired property to fail catastrophically. We have

$$\begin{aligned}
({a_n}^{b_n}) &= (2^n) \\
({a'_n}^{b_n}) &= (3^n) \\
({a_n}^{b'_n}) &= (4^n) \\
({a'_n}^{b'_n}) &= (9^n)
\end{aligned}$$

so that each of the sequences $({a_n}^{b_n}), ({a'_n}^{b_n}), ({a_n}^{b'_n}), ({a'_n}^{b'_n})$ grows more slowly than the next.

3.4. Partial ordering

Now we will formalize the notion of "size" of growth orders by defining a partial ordering $\leq$ that allows us to compare them. We will see in the Section 3.5 that this notion of inequality interacts with the previously defined operations in favorable ways.

Definition 20. Let $\mathfrak{a}, \mathfrak{b}$ be growth orders. We will say that $\mathfrak{a} \leq \mathfrak{b}$, or $\mathfrak{a}$ **grows at most as fast** as $\mathfrak{b}$, if, for each $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$, there exists a constant $C \in \mathbb{R}^+$ such that

$$a_n \leq C b_n$$

for all $n \in \mathbb{N}$. Further, we will say that $\mathfrak{a} < \mathfrak{b}$, or $\mathfrak{a}$ **grows slower** than $\mathfrak{b}$, if $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a} \neq \mathfrak{b}$.

It is straightforward to show that the above defines a partial ordering on the growth orders over $\mathcal{S}(\mathbb{R}^+)$.

Proposition 21. The above defines a partial ordering on growth orders in $\mathcal{S}(\mathbb{R}^+)/\sim$.

Proof. We immediately have the reflexive property, namely that $\mathfrak{a} \leq \mathfrak{a}$ for all $\mathfrak{a}$, for if $(a_n) \in \mathfrak{a}$, we have that $a_n \leq C a_n$ for all $n \in \mathbb{N}$ when $C = 1$.

To prove transitivity, let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be growth orders with $\mathfrak{a} \leq \mathfrak{b} \leq \mathfrak{c}$. If $(a_n) \in \mathfrak{a}, (b_n) \in \mathfrak{b}$, and $(c_n) \in \mathfrak{c}$, then there exist constants C_1, C_2 such that $a_n \leq C_1 b_n$ and $b_n \leq C_2 c_n$, and therefore $a_n \leq C_1 C_2 c_n$ for all $n \in \mathbb{N}$.

Finally, we shall prove antisymmetry: namely that $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{a}$ together imply that $\mathfrak{a} = \mathfrak{b}$. If both of these inequalities hold, then for all $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$, there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that $a_n \leq C_1 b_n$ and $b_n \leq C_2 a_n$, meaning that

$$C_1^{-1} a_n \leq b_n \leq C_2 a_n$$

and therefore $(a_n) \sim (b_n)$ and $\mathfrak{a} = \mathfrak{b}$. ◻

Note that if we want to claim that $[\alpha] \leq [\beta]$ where $\alpha = (a_n)$ and $\beta = (b_n)$, with the propositions we have proven so far and our current definition of $\leq$, it would *not be sufficient* to find a constant $C > 0$ such that $a_n \leq C b_n$ for all $n \in \mathbb{N}$. This is because the definition requires such an inequality to hold for *any pair* of sequences with the growth orders $[\alpha]$ and $[\beta]$. We will now show that it is, in fact, sufficient to establish the inequality for a specific pair of sequences.

Proposition 22. For any two sequences $\alpha = (a_n)$ and $\beta = (b_n)$ in $\mathcal{S}(\mathbb{R}^+)$, we have that $[\alpha] \leq [\beta]$ iff there exists $C > 0$ such that $a_n \leq C b_n$ for all $n \in \mathbb{N}$.

Proof. The "only if" direction is trivial, by the definition of inequality of growth orders. So we need only consider the "if" direction.

Suppose that $C > 0$ is such that $a_n \leq C b_n$ for all $n \in \mathbb{N}$, and let $(a'_n) \sim (a_n)$ and $(b'_n) \sim (b_n)$ be arbitrary sequences of the same respective growth orders. By the definition of $\sim$, there exists $C_1 > 0$ such that $a'_n \leq C_1 a_n$ for all n, and similarly there exists $C_2 > 0$ such that $b'_n \geq C_2 b_n$ for all n. This implies that $a'_n \leq C_1 C_2^{-1} C b'_n$, hence $a'_n \leq C' b'_n$ for all n, where $C' = C_1 C_2^{-1} C > 0$. Since $(a'_n) \in [\alpha]$ and $(b'_n) \in [\beta]$ were arbitrary, we have $[\alpha] \leq [\beta]$ by definition, which proves our claim. ◻

The following proposition establishes some additional equivalent conditions for inequality of two growth orders. Their proof is trivial, so we omit it.

Proposition 23. The following are equivalent to $[\alpha] \leq [\beta]$:

- $a_n = \mathcal{O}(b_n)$
- a_n / b_n is bounded
- $a_n / b_n \leq (1)$

Proof. Follows directly from definitions. ◻

We have defined a partial ordering on $\mathcal{S}(\mathbb{R}^+)/\sim$ (so that it can be called a *poset*), but it is *not* a total order. That is, trichotomy does not hold, and there exist growth orders $\mathfrak{a}$ and $\mathfrak{b}$ such that neither $\mathfrak{a} \leq \mathfrak{b}$ not $\mathfrak{b} \leq \mathfrak{a}$. For instance, consider $\mathfrak{a} = \mathbf{1}$ and $\mathfrak{b} = \mathfrak{u}$.

Definition 24. If $\mathfrak{a} \not\leq \mathfrak{b}$ and $\mathfrak{b} \not\leq \mathfrak{a}$, then we say that $\mathfrak{a}$ is **incomparable** or **not comparable** to $\mathfrak{b}$, and write $\mathfrak{a} \perp \mathfrak{b}$. On the other hand, we say that $\mathfrak{a}$ and $\mathfrak{b}$ are **comparable** if either $\mathfrak{a} \leq \mathfrak{b}$ or $\mathfrak{b} \leq \mathfrak{a}$. A **chain** is a set of growth orders of which any two are comparable, and an **antichain** is a set of growth orders of which any two are incomparable.

Later on, we will try to construct *chains* of growth orders. Pairwise comparability of growth orders is convenient for arithmetic. Another strong motivation for working with chains of growth orders comes from the application of asymptotics to the analysis of algorithms in computer science, where the growth order of the resource consumption of an algorithm (e.g. in units of time or computer memory) is used as a kind of metric for ranking it in relation to other algorithms. To be able to compare algorithms like this, it is necessary that growth orders arising this way be pairwise comparable. However, it is harder than it seems to concisely describe a way of restricting $\mathcal{S}(\mathbb{R}^+)/\sim$ to a subset of growth orders that is both closed under desirable operations like multiplication (and later, in Section 4.1, partial summation) while still possessing trichotomy.

We may also study the *order type* of a chain of growth orders. Briefly, an order type is an equivalence class of total orderings that are pairwise isomorphic to each other, where two order types are "isomorphic" if there exists an order-preserving bijection between them - intuitively, if they have the "same structure up to relabeling". For a detailed treatment of order types and their arithmetic, see [7]. Here are a few fairly well-behaved chains of $\mathcal{S}(\mathbb{R}^+)/\sim$ that are all closed under multiplication, as well as their order types (according to Sierpinski's naming system):

- The set of polynomial growth orders $\mathfrak{n}^p$ with $p \in \mathbb{N}$, which has order type ω (the order type of $\mathbb{N}$).
- The set of power-function growth orders $\mathfrak{n}^p$ with $p \in \mathbb{R}$, which has order type λ (the order type of $\mathbb{R}$).
- The set of growth orders taking the form $\mathfrak{n}^p\mathfrak{l}^q = [(n^p \log^q n)]$ with $p, q \in \mathbb{R}$. This has order type λ^2.

In set theory, the ordinal numbers are sometimes used to quantify the "size" or "depth" of an order type. In particular, each order type is assigned the smallest ordinal number that cannot be embedded in it, whose existence is guaranteed by the fact that the ordinals are well-ordered and unbounded in cardinality. This gives rise to the following question about the "size" of the partially ordered structure that we have just defined:

Question 1 Which ordinal numbers are the order type of some chain of $\mathcal{S}(\mathbb{R}^+)/\sim$? That is, what is the smallest ordinal that *cannot* be embedded in $\mathcal{S}(\mathbb{R}^+)/\sim$?

It happens that $\mathcal{S}(\mathbb{R}^+)/\sim$ also has some very large antichains. For instance, consider the family of growth orders

$$\mathfrak{u}_p = n^{p(-1)^n}$$

where $p \in \mathbb{R}$. We can see that $\mathfrak{u}_p \perp \mathfrak{u}_q$ for all $p \neq q$, since the ratio of the terms $n^{p(-1)^n}$ and $n^{q(-1)^n}$ will oscillate between very large and very small values. The above observations prove the following proposition:

Proposition 25. $\mathcal{S}(\mathbb{R}^+)/\sim$ has both uncountable chains and uncountable antichains.

3.5. Inequalities

We will now study how the operations defined in the previous sections interact with the partial ordering defined on growth orders. The following proposition tells us that comparing growth orders $\mathfrak{a}$ and $\mathfrak{b}$ is the same as comparing their quotient $\mathfrak{a}/\mathfrak{b}$ to $\mathbf{1}$ - a fact that we will make use of often in sections to come.

Proposition 26. We have $\mathfrak{a} \leq \mathfrak{b}$ iff $\mathfrak{a}/\mathfrak{b} \leq \mathbf{1}$, and $\mathfrak{a} \perp \mathfrak{b}$ iff $\mathfrak{a}/\mathfrak{b} \perp \mathbf{1}$.

Proof. Follows straight from the definitions. Because we are dealing with sequences of positive real numbers, we have that $a_n \leq Cb_n$ iff $a_n/b_n \leq C \cdot 1$, from which $\mathfrak{a} \leq \mathfrak{b} \iff \mathfrak{a}/\mathfrak{b} \leq \mathbf{1}$ immediately follows. Since $\mathfrak{a} \perp \mathfrak{b}$ iff neither $\mathfrak{a} \leq \mathfrak{b}$ nor $\mathfrak{b} \leq \mathfrak{a}$, we have that $\mathfrak{a} \perp \mathfrak{b} \iff \mathfrak{a}/\mathfrak{b} \perp \mathbf{1}$ follows immediately. □

A useful property of reciprocals is that they *reverse* the ordering of growth orders:

Proposition 27. If $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{b}^{-1} \leq \mathfrak{a}^{-1}$.

Proof. If $\mathfrak{a} \leq \mathfrak{b}$, then $a_n \leq Cb_n$ for some $C \in \mathbb{R}^+$ by definition, and since a_n, b_n are positive for all $n \in \mathbb{N}$, we have by dividing both sides by $a_n b_n$ that $b_n^{-1} \leq C a_n^{-1}$. This shows that $\mathfrak{b}^{-1} \leq \mathfrak{a}^{-1}$ by definition. □

For comparable growth orders, addition behaves like a "maximum" function:

Proposition 28. If $\mathfrak{a}, \mathfrak{b}$ are comparable, then $\mathfrak{a} + \mathfrak{b} = \max(\mathfrak{a}, \mathfrak{b})$.

Proof. Suppose WLOG that $\mathfrak{a} \geq \mathfrak{b}$. If $\alpha = (a_n) \in \mathfrak{a}$ and $\beta = (b_n) \in \mathfrak{b}$, we have that there exists a constant $C \in \mathbb{R}^+$ such that $a_n \geq Cb_n$ for all $n \in \mathbb{N}$, implying that $a_n \geq \frac{C}{C+1}(a_n + b_n)$ and therefore $\mathfrak{a} \geq \mathfrak{a} + \mathfrak{b}$. On the other hand, we have $a_n \leq a_n + b_n$, so $\mathfrak{a} \leq \mathfrak{a} + \mathfrak{b}$, and therefore $\mathfrak{a} = \max(\mathfrak{a}, \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}$. □

This might give the impression that $+$ is a rather uninteresting operation on growth orders. However, the above only applies to comparable growth orders: the situation is more complicated (and more interesting!) for incomparable growth orders $\mathfrak{a} \perp \mathfrak{b}$. We will now use this operation to prove that the order structure of $\mathcal{S}(\mathbb{R}^+)/\sim$ is that of a *lattice*, or an ordered set in which each pair of elements has a unique least upper bound and greatest lower bound.

Proposition 29. The set of growth orders $\mathcal{S}(\mathbb{R}^+)/\sim$ comprises a lattice in which the **join** and **meet** are respectively defined by

$$\mathfrak{a} \vee \mathfrak{b} = \mathfrak{a} + \mathfrak{b}$$

$$\mathfrak{a} \wedge \mathfrak{b} = (\mathfrak{a}^{-1} + \mathfrak{b}^{-1})^{-1}$$

so that $\mathfrak{a} \vee \mathfrak{b}$ is the unique least upper bound of $\mathfrak{a}, \mathfrak{b}$, and $\mathfrak{a} \wedge \mathfrak{b}$ is their unique greatest lower bound.

Proof. First we prove that $\mathfrak{a} \vee \mathfrak{b}$ is the unique least upper bound of $\mathfrak{a}$ and $\mathfrak{b}$. Suppose that $\mathfrak{c} \geq \mathfrak{a}, \mathfrak{b}$, so that for all $(a_n) \in \mathfrak{a}, (b_n) \in \mathfrak{b}, (c_n) \in \mathfrak{c}$, we have constants $C_1, C_2 \in \mathbb{R}^+$ such that $a_n \leq C_1 c_n$ and $b_n \leq C_2 c_n$ for all $n \in \mathbb{N}$. Then $a_n + b_n \leq (C_1 + C_2) c_n$, meaning that $\mathfrak{a} + \mathfrak{b} \leq \mathfrak{c}$. Hence $\mathfrak{a} + \mathfrak{b}$ is a least upper bound for $\mathfrak{a}, \mathfrak{b}$ because any other common upper bound $\mathfrak{c}$ must grow at least as fast as it does. Uniqueness follows from antisymmetry of $\leq$: if there were two least upper bounds $\mathfrak{c}_1, \mathfrak{c}_2$, we would have that $\mathfrak{c}_1 \leq \mathfrak{c}_2$ and $\mathfrak{c}_2 \leq \mathfrak{c}_1$, and therefore $\mathfrak{c}_1 = \mathfrak{c}_2$.

To show that $\mathfrak{a} \wedge \mathfrak{b}$ is the unique greatest lower bound, notice that $\mathfrak{c}$ is a lower bound for $\mathfrak{a}, \mathfrak{b}$ if and only if $\mathfrak{c}^{-1}$ is an upper bound for $\mathfrak{a}^{-1}, \mathfrak{b}^{-1}$ because the reciprocal function $\cdot \mapsto \cdot^{-1}$ is a decreasing bijection. (Decreasingness is proven in Proposition 27, and bijectivity follows from the fact that it is its own inverse.) Hence, the greatest-lower-bound property of $\mathfrak{a} \wedge \mathfrak{b}$, as well as its uniqueness, is a corollary of least-upper-bound property of $\mathfrak{a} \vee \mathfrak{b}$ combined with the fact that $\mathfrak{a} \vee \mathfrak{b} = (\mathfrak{a}^{-1} \wedge \mathfrak{b}^{-1})^{-1}$. □

Proposition 29 implies that every pair of growth orders has a *least upper bound* and a *greatest lower bound*, and consequently that any *finite* collection of growth orders has a LUB and a GLB (which can be formed by repeatedly taking pairwise LUBs and GLBs). A natural question to ask is whether *arbitrary* bounded sets of growth orders also have unique least upper bounds and greatest lower bounds. However, the question of whether least upper bounds exist in $\mathcal{S}(\mathbb{R}^+)$ can be answered in the negative fairly quickly. Consider, for instance, the chain

$$\mathfrak{n} < \mathfrak{n}^2 < \mathfrak{n}^3 < \cdots$$

and suppose that $\mathfrak{a}$ is an upper bound for the set $\{\mathfrak{n}^p\}_{p \in \mathbb{N}}$. No matter the value of $\mathfrak{a}$, there always exists a smaller upper bound for this chain. For instance, $\mathfrak{a}/\mathfrak{n}$ will suffice: if $\mathfrak{a} > \mathfrak{n}^p$ for all $p \in \mathbb{N}$, then $\mathfrak{a}/\mathfrak{n} > \mathfrak{n}^p$ for all $p \in \mathbb{N}$ as well, yet $\mathfrak{a}/\mathfrak{n} < \mathfrak{a}$.

Thus, we cannot even get least upper bounds for increasing sequences of growth orders in $\mathcal{S}(\mathbb{R}^+)$. A natural follow-up question is whether *any* strictly increasing sequence of growth orders has a least upper bound. Or if we have an increasing sequence of growth orders

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots$$

can we *always* conclude that no upper bound is a least upper bound, as was the case with the chain $\mathfrak{n} < \mathfrak{n}^2 < \cdots$? The following proposition shows that the latter is true.

Proposition 30. For any strictly increasing sequence of growth orders

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots$$

with an upper bound $\mathfrak{a}' > \mathfrak{a}_i$ for all $i \in \mathbb{N}$, then there exists another upper bound $\mathfrak{b}$ with $\mathfrak{b} > \mathfrak{a}_i$ for all $i \in \mathbb{N}$ and $\mathfrak{b} < \mathfrak{a}'$.

Proof. Making use of the Axiom of Choice, we may consider some infinite sequence of sequences $(a_n^{(i)}) \in \mathfrak{a}_i$ for $i \in \mathbb{N}$. Without loss of generality, we may assume that $a_n^{(i)} \leq a_n^{(j)}$ for all $i < j$ and all $n \in \mathbb{N}$. For if the sequences we choose do not satisfy these inequalities, we may let C_i be a family of constants such that $a_n^{(i)} \leq C_i a_n^{(i+1)}$ for all $n \in \mathbb{N}$ (since $\mathfrak{a}_i < \mathfrak{a}_{i+1}$) and replace the sequences $(a_n^{(1)}), (a_n^{(2)}), (a_n^{(3)}), \cdots$ with the sequences $(a_n^{(1)}), (C_1 a_n^{(2)}), (C_1 C_2 a_n^{(3)}), \cdots$, which have the same respective growth orders while satisfying the desired inequalities.

Having chosen a sequence of sequences $(a_n^{(i)})$ with $a_n^{(i)} \leq a_n^{(j)}$ for all $i < j$ and $n \in \mathbb{N}$, let us now consider an arbitrary sequence $(a'_n) \in \mathfrak{a}'$. Since $\mathfrak{a}' > \mathfrak{a}_i$ for all $i \in \mathbb{N}$, we have that for any fixed $i \in \mathbb{N}$, the sequence of ratios $a'_n / a_n^{(i)}$ is unbounded above. We may therefore define a sequence of indices (m_i) as follows: let $m_1 = 1$, and let m_{i+1} be the smallest natural number strictly greater than m_i such that $a'_m / a_m^{(i)} \geq i$.

We are now ready to use a "diagonalization" technique to define a sequence (b_n) with an intermediate growth order. Define the sequence as follows:

$$b_n = \begin{cases} a_n^{(i)} & \text{if } n = m_i,\ i \in \mathbb{N} \\ a'_n & \text{else} \end{cases}$$

We can show that if $\mathfrak{b} = [(b_n)]$, then $\mathfrak{b} < \mathfrak{a}'$ while $\mathfrak{b} > \mathfrak{a}_i$ for all $i \in \mathbb{N}$. First of all: for any fixed $i \in \mathbb{N}$, we have for all $n > m_i$ that $b_n \geq a_n^{(i+1)}$ (which can be seen easily by considering the two cases in the definition of b_n) and therefore $\mathfrak{b} \geq \mathfrak{a}_{i+1} > \mathfrak{a}_i$. Secondly, we may deduce that $\mathfrak{b} < \mathfrak{a}'$ by considering the ratio b_n / a'_n. For any $n \in \mathbb{N}$, we either have that $b_n / a'_n = 1$ (when $n \neq m_i$ for any $i \in \mathbb{N}$) or $b_n / a'_n \leq 1/i \leq 1$ for some $i \in \mathbb{N}$ (when $n = m_i$, because m_i is defined such that $a_{m_i}^{(i)} / a'_{m_i} \leq 1/i$). Thus, we have that the sequence (b_n / a'_n) is bounded above by 1 but comes arbitrarily close to 0, meaning that $\mathfrak{b}/\mathfrak{a}' < \mathbf{1}$ and therefore $\mathfrak{b} < \mathfrak{a}'$. Thus, we have constructed $\mathfrak{b}$ such that

$$\mathfrak{a}_1 < \mathfrak{a}_2 < \mathfrak{a}_3 < \cdots < \mathfrak{b} < \mathfrak{a}'$$

as claimed. □

Although we have just proven that no strictly increasing sequence of growth orders in $\mathcal{S}(\mathbb{R}^+)$ has a *least upper bound*, it is in fact true that every increasing sequence of growth orders has *some* upper bound.

Proposition 31. For any chain of growth orders

$$\mathfrak{a}_1 \leq \mathfrak{a}_2 \leq \mathfrak{a}_3 \leq \cdots$$

there exists a growth order $\mathfrak{a}'$ such that $\mathfrak{a}' \geq \mathfrak{a}_i$ for all $i \in \mathbb{N}$.

Proof. We can complete this proof using a diagonalization argument. Let us choose one sequence from each growth order $(a_n^{(i)}) \in \mathfrak{a}_i$ (making use of the Axiom of Choice). Then we may define a sequence (a_n') as follows:

$$a_n' = \sup_{1 \leq i \leq n} a_n^{(i)}$$

so that $a_n' \geq a_n^{(i)}$ for all $n \geq i$, for all $i \in \mathbb{N}$. This means that if $\mathfrak{a}' = [(a_n')]$, we have that $\mathfrak{a}' \geq \mathfrak{a}_i$ for all $i \in \mathbb{N}$, as desired. □

We have seen that the ordering on $\mathcal{S}(\mathbb{R}^+)$ differs from the ordering on, say, $\mathbb{R}^+$ in several key ways: for one, bounded sequences in $\mathbb{R}^+$ always have least upper bounds, which is not true in this poset; on the other hand, not all sequences in $\mathbb{R}^+$ have a upper bound *at all*, but in this ordering all sequences are bounded. Proposition 31 proves that this is the case for all *increasing* sequences of growth orders, but even for non-increasing sequences of the form

$$\mathfrak{a}_1, \ \mathfrak{a}_2, \ \mathfrak{a}_3, \ \cdots$$

we can construct an upper bound by using 29 and considering instead the increasing sequence

$$\mathfrak{a}_1 \ \leq \ \mathfrak{a}_1 + \mathfrak{a}_2 \ \leq \ \mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 \ \leq \ \cdots$$

and similarly for a lower bound.

4. Partial summation

4.1. Definition of partial summation

In my mind, one of the principal motivations for developing an algebraic theory of growth orders was to address the following question: given a sequence of known growth order, how can we determine the growth order of its sequence of partial sums? That is, given the growth order of a sequence (a_n), are there any general rules or principles allowing us to deduce the growth order of

$$\sum_{i=1}^{n} a_i \ \sim \ ?$$

At first, the author was just as interested in finding "quick and dirty" tricks for calculating asymptotic formulas for sums that appeared, for instance, in computational complexity considerations for algorithms in computer science. The situation turned out to be more interesting and complex than expected.

Let us begin by defining this as an operation on sequences, and showing that it is well-defined as an operation on growth orders.

Definition 32. Given a sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$, define its sequence of **partial sums**, denoted $\Sigma\alpha$, as the sequence

$$\left(\sum_{i=1}^{n} a_i\right)$$

.

Proposition 33. If $\alpha \sim \alpha'$, then $\Sigma\alpha \sim \Sigma\alpha'$.

Proof. Suppose that $\alpha \sim \alpha'$, so that there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

for all $n \in \mathbb{N}$. It follows that

$$C_1 \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a'_n \leq C_2 \sum_{i=1}^{n} a_i$$

so that we have $\Sigma\alpha \sim \Sigma\alpha'$ by definition. □

Therefore, the following definition is justified:

Definition 34. Given a growth order $\mathfrak{a} = [\alpha]$, define its **partial sum** to be the growth order $\Sigma\mathfrak{a} = [\Sigma\alpha]$.

Notice that Σ is a unary operation [1] on growth orders, or $\Sigma : \mathcal{S}(\mathbb{R}^+)/\sim \longrightarrow \mathcal{S}(\mathbb{R}^+)/\sim$. This is very different from how sigma-notation works on real numbers: when expressing a sum of real numbers using sigma-notation, we must sum over an indexed sequence of real numbers and specify starting and ending indices. When taking partial sums of growth orders, however, no indexing is necessary, for the indexing is intrinsic to the sequences contained within any given growth order.

Here are some elementary properties of this new operation:

Proposition 35. The following facts hold for arbitrary growth orders $\mathfrak{a}, \mathfrak{b}$:

$$\begin{aligned} \Sigma\mathfrak{a} &\geq \mathbf{1} \\ \Sigma\mathfrak{a} &\geq \mathfrak{a} \\ \mathfrak{a} \leq \mathfrak{b} &\implies \Sigma\mathfrak{a} \leq \Sigma\mathfrak{b} \\ \Sigma(\mathfrak{a} + \mathfrak{b}) &= \Sigma\mathfrak{a} + \Sigma\mathfrak{b} \end{aligned}$$

Proof. These four identities follow from the fact that their elementwise analogues for sequences are also true:

$$\begin{aligned} \Sigma\alpha &\geq (1) \\ \Sigma\alpha &\geq \alpha \\ \alpha \leq \beta &\implies \Sigma\alpha \leq \Sigma\beta \\ \Sigma(\alpha + \beta) &= \Sigma\alpha + \Sigma\beta \end{aligned}$$

□

Notice that the inequality $\mathfrak{a} \neq \mathfrak{b}$ does not imply $\Sigma\mathfrak{a} \neq \Sigma\mathfrak{b}$ in general - that is, Σ is not injective as a function on growth orders. As a simple example, consider any two growth orders whose partial sums are convergent, such as $\mathfrak{n}^{-2}$ and $\mathfrak{n}^{-3}$ which have $\mathfrak{n}^{-2} \neq \mathfrak{n}^{-3}$ but $\Sigma\mathfrak{n}^{-2} = \Sigma\mathfrak{n}^{-3} = \mathbf{1}$. However, there are also examples with divergent partial sums: for example, consider the sequences $\mathfrak{n}$ and $\mathfrak{u}$, which are unequal despite the fact that their partial sums have the same growth order $\Sigma\mathfrak{n} = \Sigma\mathfrak{u} = \mathfrak{n}^2$.

At this point, we might wonder whether this problem only arises when $\mathfrak{a} \perp \mathfrak{b}$. That is, if $\mathfrak{a} \neq \mathfrak{b}$ *and* $\mathfrak{a}, \mathfrak{b}$ are comparable, then perhaps from this we can deduce that $\Sigma\mathfrak{a} \neq \Sigma\mathfrak{b}$? Alas, this also fails to be true. As a counterexample, consider $\mathfrak{a} = \mathbf{1}$ and $\mathfrak{b}$ equal to the growth order of the sequence $\beta = (b_n)$ defined piecewise as follows:

[1] That is, a function of one argument.

$$b_n = \begin{cases} k & \text{if } n = 2^k \\ 1 & \text{else} \end{cases}$$

In this case, we have $\mathfrak{a} < \mathfrak{b}$ because $\mathfrak{b}$ is bounded below by 1 yet is unbounded, and the entries of $\Sigma\beta$ are $n + O(\log^2 n)$, meaning that $\Sigma\mathfrak{a} = \Sigma\mathfrak{b}$. In Section 4.2, after introducing the idea of a *monotone growth order*, we will determine a sufficient criterion on growth orders $\mathfrak{a} < \mathfrak{b}$ that guarantee $\Sigma\mathfrak{a} < \Sigma\mathfrak{b}$.

With a bit of effort, we may prove that much like the operations defined earlier, the partial sum operation preserves moderate growth.

Proposition 36. If $\mathfrak{a}$ is moderate, then $\Sigma\mathfrak{a}$ is moderate.

Proof. Let $k \in \mathbb{N}$ be given. By the moderateness of $\mathfrak{a}$, for any $\alpha = (a_n) \in \mathfrak{a}$, there exist constants C_1, C_2 such that for all m with $n \leq m \leq kn$, we have

$$C_1 a_n \leq a_m \leq C_2 a_n$$

Now let us fix some $m, n \in \mathbb{N}$ with $n \leq m \leq kn$, and consider the sum

$$\sum_{i=1}^{m} a_i$$

Because the a_i are positive and $m \geq n$, we have that

$$\sum_{i=1}^{m} a_i \geq \sum_{i=1}^{n} a_i$$

On the other hand, we have that

$$\begin{aligned} \sum_{i=1}^{m} a_i &\leq \sum_{i=1}^{kn} a_i \\ &= \sum_{j=0}^{n-1} \sum_{i=1}^{k} a_{jk+i} \\ &\leq \sum_{j=0}^{n-1} \sum_{i=1}^{k} C_2 a_{jk+1} \end{aligned}$$

because $jk + 1 \leq jk + i \leq k(jk + 1)$ for all $k \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$, and $i \in \{1, \cdots, k\}$. We may further simplify this upper bound as follows:

$$\begin{aligned}\sum_{i=1}^{m} a_i &\leq \sum_{j=0}^{n-1} kC_2 a_{jk+1} \\ &\leq \sum_{j=0}^{n-1} kC_2^2 a_{j+1} \\ &= kC_2^2 \sum_{i=1}^{n} a_i\end{aligned}$$

since $j+1 \leq jk+1 \leq k(j+1)$ for all $j \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$. Thus, the sum can be bounded both above and below as follows:

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{m} a_i \leq kC_2^2 \sum_{i=1}^{n} a_i$$

which proves that $\Sigma\alpha$ has moderate growth by definition, and that $\Sigma\mathfrak{a}$ is moderate as claimed. □

The below proposition shows that taking partial sums of a moderate growth order increases its growth order by at least a factor of $\mathfrak{n}$. In Section 4.3, we will study in more detail the factor by which taking partial sums can increase a growth order.

Proposition 37. If $\mathfrak{a}$ is moderate, then $\Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$.

Proof. Let $(a_n) \in \mathfrak{a}$ be moderate. From Proposition 36, we have that $\Sigma\mathfrak{a}$ is moderate, therefore

$$\left(\sum_{i=1}^{n} a_i\right) \sim \left(\sum_{i=1}^{2n} a_i\right)$$

and we may split up this sum as follows:

$$\sum_{i=1}^{2n} a_i = \sum_{i=1}^{n} a_i + \sum_{i=n+1}^{2n} a_i$$

From Proposition 12, we have that this is $\sim \Sigma(a_n) + (na_n)$, which grows at least as fast as (na_n), with growth order $\mathfrak{n}\mathfrak{a}$. Thus, we have that $\Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$ as claimed. □

4.2. Monotone growth orders

Now we will introduce another "niceness" condition on growth orders, akin to moderateness.

Definition 38. We say that a growth order $\mathfrak{a}$ is **monotone** if it contains some monotone sequence $\alpha = (a_n) \in \mathfrak{a}$. If it contains a monotone increasing sequence, we may call it **monotone increasing**, and if it contains a monotone decreasing sequence, we may call it **monotone decreasing**.

Unlike moderateness, the monotonicity condition does not apply to *all* sequences of a given growth order. Rather, if $\mathfrak{a}$ contains *some* monotone sequence, it is called a monotone growth order, even though it will contain many non-monotone sequences as well.

Proposition 39. The only growth order that is both monotone increasing and monotone decreasing is the constant growth order $\mathbf{1}$.

Proof. We can see that $\mathbf{1}$ is both monotone increasing and decreasing, because the constant sequence $(1) \in \mathbf{1}$ is both a monotone increasing and a monotone decreasing sequence.

Now suppose $\mathfrak{a}$ is both monotone increasing and monotone decreasing. Then let $(a_n) \in \mathfrak{a}$ be monotone increasing, and let $(a'_n) \in \mathfrak{a}$ be monotone decreasing. Since (a_n) and (a'_n) have the same growth order, we have a_n/a'_n bounded above by some constant $C > 0$, so that $a_n \le Ca'_n \le Ca'_1$ for all $n \in \mathbb{N}$, by the monotone decreasingness of (a'_n). This is a constant upper bound, implying that $\mathfrak{a} \le \mathbf{1}$. On the other hand, we have that a'_n/a_n is bounded below by some $D > 0$, so that $a'_n \ge Da_n \ge Da_1$ by monotone increasingness of (a_n). This is a positive constant lower bound, implying that $\mathfrak{a} \ge \mathbf{1}$. Hence, since $\mathfrak{a} \ge \mathbf{1}$ and $\mathfrak{a} \le \mathbf{1}$, we have that $\mathfrak{a} = \mathbf{1}$ as claimed. □

The following simple proposition lies at the heart of the usefulness of the monotonicity condition:

Proposition 40. If $\mathfrak{a}$ is monotone, then it is comparable to $\mathbf{1}$.

Proof. Suppose that $\alpha = (a_n) \in \mathfrak{a}$ is a monotone sequence. If (a_n) is monotone increasing, then it is bounded below by a_1, and therefore $\mathfrak{a} \ge \mathbf{1}$. If it is monotone decreasing, then it is bounded above by a_1, and we have that $\mathfrak{a} \le \mathbf{1}$. □

Why is this significant? Much of the unusual/edge-case behavior that we have seen in previous counterexamples arose from the existence of incomparable growth orders, and their relationships to each other. Monotonicity, however, guarantees that a growth order is comparable to constant growth, meaning that if $\mathfrak{a}, \mathfrak{b}$ are such that their *quotient* $\mathfrak{a}/\mathfrak{b}$ is monotone, then this quotient must be comparable to $\mathbf{1}$, meaning that $\mathfrak{a}$ is comparable to $\mathfrak{b}$. This *monotone-quotient property* therefore guarantees comparability (and several other useful things besides, as we shall see in Sections 4.3 and 5.5), so in Section 6.1 we will explore ways of constructing large collections of growth orders whose pairwise quotients are monotone, in order to guarantee that any two of them are comparable.

A natural question arises from the Proposition 40: if all monotone growth orders are comparable to $\mathbf{1}$, might it be the case that all monotone growth orders are comparable *amongst themselves*? Unfortunately, this is not the case. As a counterexample, consider the growth orders $\mathfrak{a} = \mathfrak{n}^{1/2}$ and $\mathfrak{b}$ equal to the growth order of the sequence $\beta = (b_n)$ defined by $b_1 = 1$ and

$$b_n = 2^{3^{\lfloor \log_3 \log_2 n \rfloor}} = \exp_2 \exp_3 \lfloor \log_3 \log_2 n \rfloor$$

for all $n \geq 2$. Clearly $\mathfrak{a}$ is monotone, and $\mathfrak{b}$ is monotone because each of the functions $\exp_2, \exp_3, \lfloor \cdot \rfloor, \log_2, \log_3$ is monotone and because $b_1 = 1 < b_2 = 2$. Notice that when $n = 2^{3^k}$ for some $k \in \mathbb{N}$, we have that $b_n = n$, whereas when $n = 2^{3^k - 1}$, we have that $b_n = (2n)^{1/3}$. Hence, if $\alpha \in \mathfrak{a}$, then α/β is unbounded on the subsequence $n = 2^{3^k - 1}$, and β/α is unbounded on the subsequence $n = 2^{3^k}$. This means that $\mathfrak{a} \perp \mathfrak{b}$ despite the fact that $\mathfrak{a}, \mathfrak{b}$ are both monotone! Apparently, monotonicity comes with some *limited* guarantees of comparability, but not all of the guarantees that we might hope for.

The guarantee of comparability that monotonicity offers is not shared by the moderateness condition: that is, moderate sequences are *not necessarily* comparable to $\mathbf{1}$. For example, consider the sequence (a_n) defined by

$$a_n = n^{\sin \log \log n}$$

for $n \geq 3$. This sequence is incomparable to $\mathbf{1}$, since it has subsequences tending to 0 and to ∞. [2] It is, however, moderate, and to see why we can use the doubling condition introduced in Proposition 7. First, notice that

$$|\log \log(2n) - \log \log n| = \left| \log \left(\frac{\log n + \log 2}{\log n} \right) \right| = \left| \log \left(1 + \frac{\log 2}{\log n} \right) \right| \leq \frac{\log 2}{\log n}$$

using the bound $\log(1 + x) \leq x$ for $x > 0$. Therefore, since $\log \log n$ is a monotone increasing function of n, we have that for all $n \leq m \leq 2n$,

$$|\log \log m - \log \log n| \leq \frac{\log 2}{\log n}$$

Now, because the sine function is Lipschitz continuous with a constant of $L = 1$, we have that

$$|\sin \log \log(m) - \sin \log \log(n)| \leq \frac{\log 2}{\log n}$$

[2] This can be seen by noticing that $\sin \log \log n$ is both above $1/2$ for infinitely many $n \in \mathbb{N}$, and below $-1/2$ for infinitely many $n \in \mathbb{N}$.

Finally, we have that

$$\begin{aligned}
&|\log(m)\cdot\sin\log\log(m)-\log(n)\cdot\sin\log\log(n)| \\
&=|\log(m/n)\cdot\sin\log\log(m)+\log(n)\cdot\sin\log\log(m)-\log(n)\cdot\sin\log\log(n)| \\
&\le|\log(m/n)\cdot\sin\log\log(m)|+|\log(n)\cdot\sin\log\log(m)-\log(n)\cdot\sin\log\log(n)| \\
&=\log(m/n)\cdot|\sin\log\log(m)|+\log(n)\cdot|\sin\log\log(m)-\sin\log\log(n)| \\
&\le\log(m/n)\cdot 1+\log(n)\cdot\frac{\log(2)}{\log(n)} \\
&\le\log(m/n)+\log(2) \\
&\le 2\log(2)
\end{aligned}$$

so that the difference

$$\log(m)\cdot\sin\log\log(m)-\log(n)\cdot\sin\log\log(n)$$

is bounded in magnitude when $n \le m \le 2n$. Now notice that

$$\frac{a_m}{a_n}=\frac{m^{\sin\log\log m}}{n^{\sin\log\log n}}=e^{\log(m)\sin\log\log(m)-\log(n)\sin\log\log(n)}$$

and since the exponent is bounded in magnitude by $2\log(2)$, the ratio a_m/a_n is bounded above by 4 and below by $1/4$. Hence, we have that $(1/4)a_n \le a_m \le 4a_n$ whenever $n \le m \le 2n$, and therefore our sequence is moderate as claimed, despite its oscillatory nature.

Hence, moderateness alone is not even sufficient to guarantee comparability to **1**, which, hopefully, allows us to appreciate why monotonicity is useful as a secondary "niceness" condition.

Question 2 What exactly is the growth order of the sum

$$\sum_{k=2}^{n} k^{\sin\log\log k} = \Theta(?)$$

Now we will begin to explore the relationship between monotonicity of growth orders and the partial summation operator Σ. One salient connection is that the monotone growth orders $> \mathbf{1}$ are precisely the growth orders $> \mathbf{1}$ that are in the image of Σ.

Proposition 41. If $\mathfrak{a} > \mathbf{1}$, then $\mathfrak{a}$ is monotone if and only if $\mathfrak{a} = \Sigma\mathfrak{b}$ for some other growth order $\mathfrak{b}$.

Proof. First, suppose $\mathfrak{a}$ is monotone, and that $\alpha = (a_n) \in \mathfrak{a}$ is a monotone sequence. It must be monotone increasing, since $\mathfrak{a} > \mathbf{1}$. If we define the sequence (b_n) by letting $b_1 = a_1 + \frac{1}{2}$ and

$$b_{n+1} = a_{n+1} - a_n + \frac{1}{2^{n+1}}$$

then we have that
$$\sum_{i=1}^{n} b_i = a_n + 1 - \frac{1}{2^n}$$
for all $n \in \mathbb{N}$, and since $\mathfrak{a} > \mathbf{1}$, we have that the constant term is negligible and $\Sigma\mathfrak{b} = \mathfrak{a}$ as desired.

The converse is is immediate, for if $\alpha = \Sigma\beta$ for some sequences $\alpha = (a_n) \in \mathfrak{a}$ and $\beta = (b_n) \in \mathfrak{b}$, then α is monotone because $a_{n+1} - a_n = b_n$ is strictly positive for all $n \in \mathbb{N}$. □

The above construction of a preimage for $\mathfrak{a}$ with respect to Σ does not necessarily respect moderateness. That is, if $\mathfrak{a}$ is moderate, the growth order $\mathfrak{b}$ constructed as above may not be moderate. The following is a question that the author has been unable to answer, but which, if answered positively, would be extremely useful for the later construction of closed chains.

Question 3 If $\mathfrak{a} > \mathbf{1}$ is both moderate and monotone, is it guaranteed that there exists a *moderate* growth order $\mathfrak{b}$ such that $\mathfrak{a} = \Sigma\mathfrak{b}$?

Now, we can extend Proposition 41 as follows:

Proposition 42. A growth order $\mathfrak{a}$ is monotone if and only if $\mathfrak{a} = \mathbf{1}$ or $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$ for some growth order $\mathfrak{b}$.

Proof. We know from the Proposition 41 that the growth orders of the form $\Sigma\mathfrak{b}$ are always monotone, meaning that those of the form $(\Sigma\mathfrak{b})^{-1}$ are monotone as well. This proves the "if" direction.

On the other hand, if $\mathfrak{a}$ is monotone, we know from a Proposition 40 that it is comparable to $\mathbf{1}$. If $\mathfrak{a} \neq \mathbf{1}$, then either $\mathfrak{a} > \mathbf{1}$, in which case there exists $\mathfrak{b}$ such that $\mathfrak{a} = \Sigma\mathfrak{b}$, or $\mathfrak{a} < \mathbf{1}$, in which case $\mathfrak{a}^{-1} > \mathbf{1}$ and there exists $\mathfrak{b}$ such that $\mathfrak{a}^{-1} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$. Thus we have proven the "only if" direction. □

Earlier, we discussed how knowing that two growth orders have a monotone quotient can be useful - in particular, it guarantees that they are comparable. Proposition 43, stated and proven below, is useful in that it shows that taking partial sums preserves the "monotone quotient property" of a pair of growth orders. We will make use of this property when constructing closed chains in Section 6.1.

Proposition 43. If $\mathfrak{a}/\mathfrak{b} > 1$ is monotone, then $\Sigma\mathfrak{a}/\Sigma\mathfrak{b}$ is monotone.

Proof. First notice that, if $\mathfrak{a}/\mathfrak{b}$ is monotone, then we can choose $(a_n) \in \mathfrak{a}, (b_n) \in \mathfrak{b}$ such that a_n/b_n is monotone. Specifically, if we choose an arbitrary monotone sequence $(r_n) \in \mathfrak{a}/\mathfrak{b}$ and and arbitrary sequence $(b_n) \in \mathfrak{b}$, then defining $(a_n) \in \mathfrak{a}$ by the equation $a_n = r_n b_n$ accomplishes this, ensuring that $(a_n/b_n) = (r_n b_n/b_n) = (r_n)$ is monotone.

Now we will make use of the elementary **mediant inequality**

$$\frac{x}{y} \le \frac{x + x'}{y + y'} \le \frac{x'}{y'}$$

which applies to all $x, x', y, y' \in \mathbb{R}^+$ with $x'/y' \ge x/y$. Consider the following inequality, which we will prove by induction for all $n \in \mathbb{N}$:

$$\frac{a_{n+1}}{b_{n+1}} \ge \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

This is true for $n = 1$ by the monotonicity of a_n/b_n, so $n = 1$ will serve as our base case. Now suppose that this inequality holds for some $n \in \mathbb{N}$, and for all preceding values. By the mediant inequality and the monotonicity of a_n/b_n, we have that

$$\frac{a_{n+2}}{b_{n+2}} \ge \frac{a_{n+1}}{b_{n+1}} \ge \frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} = \frac{a_{n+1} + \sum_{i=1}^{n} a_i}{b_{n+1} + \sum_{i=1}^{n} b_i} \ge \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

This inequality establishes both of the following inequalities:

$$\frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} \ge \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

$$\frac{a_{n+2}}{b_{n+2}} \ge \frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i}$$

the former of which proves that the sequence $\Sigma\alpha/\Sigma\beta$ is monotonic up to index $n + 1$, and the latter of which extends our original assumption from case n to case $n + 1$, allowing us to inductively prove our claim for all $n \in \mathbb{N}$. □

Proposition 44. If $\mathfrak{a}/\mathfrak{b} > \mathbf{1}$ is monotone and $\Sigma\mathfrak{a} > \mathbf{1}$, then $\Sigma\mathfrak{a}/\Sigma\mathfrak{b} > \mathbf{1}$.

Proof. We may use the same construction as in Proposition 43 to choose $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$ such that α/β is monotone. Letting $R \in \mathbb{R}^+$ be arbitrary, we will show that $\Sigma\alpha/\Sigma\beta$ eventually exceeds R, and is therefore unbounded.

Because α/β is monotone and > 1, there exists $N \in \mathbb{N}$ such that $a_n/b_n > 2R$ for all $n \ge N$. Furthermore, since $\Sigma\mathfrak{a} > \mathbf{1}$, we have that $\Sigma\mathfrak{a}$ is unbounded, and there therefore exists $M \in \mathbb{N}$ such that

$$\sum_{i=N}^{M} a_i \ge 2R \sum_{i=1}^{N-1} b_i$$

Then we have the following inequality for all $K > M$:

$$\frac{\sum_{i=1}^{K} a_i}{\sum_{i=1}^{K} b_i} = \frac{\sum_{i=1}^{N-1} a_i + \sum_{i=N}^{K} a_i}{\sum_{i=1}^{N-1} b_i + \sum_{i=N}^{K} b_i} > \frac{\sum_{i=N}^{K} a_i}{\sum_{i=1}^{N-1} b_i + \sum_{i=N}^{K} b_i}$$

Now notice that the numerator of this ratio is greater than or equal to R times the denominator, since it is greater than or equal to $2R$ times each of the two sums in the denominator. Thus, we have that

$$\frac{\sum_{i=1}^{K} a_i}{\sum_{i=1}^{K} b_i} > R$$

and, since $\Sigma\alpha/\Sigma\beta$ is monotone by Proposition 43, we have that all elements of $\Sigma\alpha/\Sigma\beta$ with $n \geq K$ exceed R. Since R was arbitrary, $\Sigma\alpha/\Sigma\beta$ is both monotone and unbounded above, and therefore $> \mathbf{1}$, as claimed. □

Recall that, in Section 4.1 of this chapter, we found a troublesome counterexample in which Σ failed to preserve strict inequality of sequences. Using the two propositions above, we are now prepared to "salvage" this idea by providing sufficient conditions for $\mathfrak{a} < \mathfrak{b}$ to imply $\Sigma\mathfrak{a} < \Sigma\mathfrak{b}$.

Proposition 45. If $\mathfrak{a}/\mathfrak{b}$ is monotone and $\Sigma\mathfrak{b} > \mathbf{1}$, then $\mathfrak{a} < \mathfrak{b} \implies \Sigma\mathfrak{a} < \Sigma\mathfrak{b}$.

Proof. This follows easily from the above Propositions 43 and 44. If $\mathfrak{a} < \mathfrak{b}$, then $\mathfrak{b}/\mathfrak{a}$ is monotone (since $\mathfrak{a}/\mathfrak{b}$ is monotone by hypothesis) and it is also $> \mathbf{1}$. By Proposition 43 and Proposition 44, since $\Sigma\mathfrak{b} > \mathbf{1}$, we have that $\Sigma\mathfrak{b}/\Sigma\mathfrak{a} > \mathbf{1}$ and therefore $\Sigma\mathfrak{a} < \Sigma\mathfrak{b}$ as claimed. □

Question 4 Is Σ injective on moderate growth orders with divergent partial sums? Or do there exist two moderate growth orders $\mathfrak{a} \neq \mathfrak{b}$ with $\Sigma\mathfrak{a}, \Sigma\mathfrak{b} > \mathbf{1}$ such that $\Sigma\mathfrak{a} = \Sigma\mathfrak{b}$?

4.3. The partial sum ratio

Here are several asymptotic formulas that are familiar from analysis

$$\begin{aligned}
\Sigma\mathfrak{n}^p &= \mathfrak{n}^{p+1} \\
\Sigma\mathfrak{n}^p\mathfrak{l}^q &= \mathfrak{n}^{p+1}\mathfrak{l}^q \\
\Sigma\mathfrak{n}^{-1} &= \mathfrak{l} \\
\Sigma\mathfrak{n}^{-1}\mathfrak{l}^p &= \mathfrak{l}^{p+1} \\
\Sigma(\mathfrak{n}\mathfrak{l})^{-1} &= \mathfrak{l}_2
\end{aligned}$$

for $p \in (-1, \infty)$ and $q \in \mathbb{R}$. If we look for patterns or tricks that might allow us to quickly calculate the asymptotics of a sequence of partial sums, the first thing that pops out is that, for a broad class of growth orders, taking partial sums amounts to just multiplying the original growth order by $\mathfrak{n}$. This is the case for the first two classes of growth orders: Σ sends $\mathfrak{n}^p$ to $\mathfrak{n} \cdot \mathfrak{n}^p = \mathfrak{n}^{p+1}$ and sends $\mathfrak{n}^p\mathfrak{l}^q$ to $\mathfrak{n} \cdot \mathfrak{n}^p\mathfrak{l}^q = \mathfrak{n}^{p+1}\mathfrak{l}^q$. However, for the growth order $\mathfrak{n}^{-1}$, taking partial sums increases the original growth order by a factor of $\mathfrak{n}\mathfrak{l}$, rather than $\mathfrak{n}$. And for the growth order $(\mathfrak{n}\mathfrak{l})^{-1}$, taking partial sums increases it by a factor of $\mathfrak{n}\mathfrak{l}\mathfrak{l}_2$.

While these examples do not suggest any obvious catch-all technique for determining the growth order of $\Sigma\mathfrak{a}$ in general (despite some noticeable patterns for special cases like $\mathfrak{n}^p\mathfrak{l}^q$), they do hint that it may be informative to study the *factor by which a growth order increases* when we take its partial sums. That is, we should take a closer look not just at the Σ function, but at the function which sends $\mathfrak{a} \to \mathfrak{a}/\Sigma\mathfrak{a}$.

Definition 46. Given any growth order $\mathfrak{a}$, let P denote $\mathfrak{a}/\Sigma\mathfrak{a}$. This quantity will be called the **partial sum ratio** of $\mathfrak{a}$.

From our observations above, we know, for instance, that $\mathrm{P}\mathfrak{a} = \mathfrak{n}^{-1}$ when $\mathfrak{a}$ takes the form $\mathfrak{n}^p\mathfrak{l}^q$ with $p > -1$, and that $\mathrm{P}\mathfrak{a} = (\mathfrak{n}\mathfrak{l})^{-1}$ when $\mathfrak{a}$ takes the form $\mathfrak{n}^{-1}\mathfrak{l}^p$ with $p > -1$. It seems that P is constant for large swaths of growth orders.

The below Proposition 47 shows that for sequences with monotone ratios, the partial sum ratio P preserves their order. This transformation is not necessarily *strictly* order-preserving on such sequences, because, as we just noticed, it maps many different growth orders to the same ratio.

Proposition 47. If $\mathfrak{a} \le \mathfrak{b}$ and $\mathfrak{a}/\mathfrak{b}$ is monotone, then $\mathrm{P}\mathfrak{a} \le \mathrm{P}\mathfrak{b}$.

Proof. Suppose $\mathfrak{a} \le \mathfrak{b}$ and $\mathfrak{a}/\mathfrak{b}$ is monotone. Since $\mathfrak{a} \le \mathfrak{b}$, we must also have $\mathfrak{a}/\mathfrak{b} \le \mathbf{1}$. If strict inequality holds, then $\mathfrak{a}/\mathfrak{b}$ must be monotone decreasing by the contrapositive of Proposition 39, and when equality holds it is both, so that in either case it is monotone decreasing. Thus there exists a monotone decreasing sequence $(c_n) \in \mathfrak{a}/\mathfrak{b}$, and we may therefore choose $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ such that $a_n/b_n = c_n$. (Let (b_n) be an arbitrary element of $\mathfrak{b}$ and let (a_n) be defined by $a_n = b_n c_n$.) Then we have that

$$c_n \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} c_n b_k \le \sum_{k=1}^{n} c_k b_k = \sum_{k=1}^{n} a_k$$

so we have that $\mathfrak{c}\Sigma\mathfrak{b} = (\mathfrak{a}/\mathfrak{b})\Sigma\mathfrak{b} \le \Sigma\mathfrak{a}$, which is equivalent to $\mathrm{P}\mathfrak{a} \le \mathrm{P}\mathfrak{b}$. $\square$

This proposition is simple but powerful, as it allows us to deduce the growth orders of the partial sums of many new sequences under only mild assumptions using a "squeezing" argument. For example, we already know that $\mathrm{P}\mathbf{1} = \mathrm{P}\mathfrak{n} = \mathfrak{n}^{-1}$, so Proposition 47 implies that if $\mathfrak{a}$ is an arbitrary growth order such that $\mathfrak{a}/\mathbf{1}$ and $\mathfrak{n}/\mathfrak{a}$ are monotone increasing, then we can instantly deduce that $\mathrm{P}\mathfrak{a} = \mathfrak{n}^{-1}$, hence $\Sigma\mathfrak{a} = \mathfrak{n}\mathfrak{a}$. We can strengthen this further by recalling that $\mathrm{P}\mathfrak{n}^p = \mathfrak{n}^{-1}$ for any exponent $p > -1$, allowing us to weaken this condition and merely require that $\mathfrak{a}/\mathfrak{n}^p$ and $\mathfrak{n}^q/\mathfrak{a}$ both be monotone increasing for *some* $p, q > -1$. These are very weak hypotheses for concluding that $\Sigma\mathfrak{a} = \mathfrak{n}\mathfrak{a}$!

The implication $\mathfrak{a} \le \mathfrak{b} \implies \mathrm{P}\mathfrak{a} \le \mathrm{P}\mathfrak{b}$ may seem to hint that P is a monotone increasing function on growth orders. However, the additional stipulation that $\mathfrak{a}/\mathfrak{b}$ be monotone is essential. Consider the two growth orders $\mathfrak{n}^{-1/2}$ and $\mathfrak{u}^{1/3}$, which satisfy $\mathfrak{n}^{-1/2} < \mathfrak{u}^{1/3}$. We also have $\Sigma\mathfrak{n}^{1/2} = \mathfrak{n}^{3/2}$ and $\Sigma\mathfrak{u}^{1/3} = \mathfrak{n}^{4/3}$, so that $\mathrm{P}\mathfrak{n}^{-1/2} = \mathfrak{n}^{-1}$ and $\mathrm{P}\mathfrak{u}^{1/3} = \mathfrak{u}^{1/3}\mathfrak{n}^{-4/3}$, so that $\mathrm{P}\mathfrak{u}^{1/3} < \mathrm{P}\mathfrak{n}^{-1/2}$. Thus, not only does P fail to be monotone increasing in general, but it actually *reverses* the order of some growth orders with a non-monotone quotient, such as $\mathfrak{n}^{-1/2} < \mathfrak{u}^{1/3}$ with $\mathrm{P}\mathfrak{u}^{1/3} < \mathrm{P}\mathfrak{n}^{-1/2}$.

4.4. Some cookbook formulas

Now we will briefly prove a few general "cookbook-style" summation formulas that will be useful to us in later sections.

Proposition 48. For any moderate growth order $\mathfrak{a}$ with $(a_n) \in \mathfrak{a}$, we have that $\Sigma P\mathfrak{a}$ is the growth order of the sequence (b_n) defined by

$$b_n = \log\left(1 + \sum_{i=1}^{n} a_i\right)$$

Proof. From the above definition of b_n, we have

$$b_{n+1} - b_n = \log\left(1 + \sum_{i=1}^{n+1} a_i\right) - \log\left(1 + \sum_{i=1}^{n} a_i\right) = \log\left(1 + \frac{a_{n+1}}{1 + \sum_{i=1}^{n} a_i}\right)$$

and by Proposition 37 we have that the ratio inside of the logarithm on the RHS decays at least as fast as $\mathfrak{n}^{-1}$. We know from analysis that $\log(1+h)$ is $\Theta(h)$ as $h \longrightarrow 0$, so we have that

$$b_{n+1} - b_n \sim \frac{a_{n+1}}{1 + \sum_{i=1}^{n} a_i}$$

The sequence with terms given by the RHS [3] of this asymptotic equivalence has growth order $\mathfrak{a}/\Sigma\mathfrak{a}$ (because $\mathfrak{a}$ is moderate). The partial sums of the RHS yield the original sequence (b_n), so we have that $\Sigma(\mathfrak{a}/\Sigma\mathfrak{a})$, or $\Sigma P\mathfrak{a}$, is the growth order of (b_n), as claimed. □

Proposition 49. For any moderate growth order $\mathfrak{a}$ with $(a_n) \in \mathfrak{a}$, and for any $p > -1$, we have that $\Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p) = (\Sigma\mathfrak{a})^{p+1}$, and for any $p < -1$, we have $\Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p) = \mathbf{1}$.

Proof. Let $p > -1$, and define a sequence (b_n) as follows:

$$b_n = \left(\sum_{i=1}^{n} a_i\right)^{p+1}$$

Using the same technique as the previous proof, we have that

$$b_{n+1} - b_n = \left(a_{n+1} + \sum_{i=1}^{n} a_i\right)^{p+1} - \left(\sum_{i=1}^{n} a_i\right)^{p+1} = \left(\sum_{i=1}^{n} a_i\right)^{p+1}\left(\left(1 + \frac{a_{n+1}}{\sum_{i=1}^{n} a_i}\right)^{p+1} - 1\right)$$

Now recall from elementary analysis that $(1+h)^{p+1} - 1$ is $\Theta(h)$ as $h \longrightarrow 0$, meaning that

$$b_{n+1} - b_n \sim \left(\sum_{i=1}^{n} a_i\right)^{p+1} \cdot \frac{a_{n+1}}{\sum_{i=1}^{n} a_i} = a_{n+1}\left(\sum_{i=1}^{n} a_i\right)^{p}$$

[3] "Right-hand side". We will also use LHS as an abbreviation for "left-hand side".

which has a growth order of $\mathfrak{a}(\Sigma\mathfrak{a})^p$, since $a_{n+1} \sim a_n$ (because $\mathfrak{a}$ is moderate). The partial sums of the LHS yield the sequence $(b_{n+1} - b_1) \sim (b_n)$, meaning that $\mathfrak{b} = (\Sigma\mathfrak{a})^{p+1} = \Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p)$, as claimed.

Suppose, on the other hand, that $p < -1$. WLOG we may suppose as well that $a_1 > 1$, so that all partial sums of (a_n) are all > 1, and the following defines a sequence of positive real numbers:

$$b_n = 1 - \Big(\sum_{i=1}^{n} a_i\Big)^{p+1}$$

Note that (b_n) has growth order $\mathbf{1}$ because it is bounded above by 1 and monotone increasing. (This is because the function $x \mapsto 1 - x^{p+1}$ is a monotone increasing function, as $p + 1 < 0$.) Further, using the same factoring trick as in the previous case, and the fact that $(1 + h)^{p+1} - 1$ is $\Theta(h)$ as $h \longrightarrow 0$, we have that

$$b_{n+1} - b_n \sim \Big(\sum_{i=1}^{n} a_i\Big)^{p+1} \cdot \frac{a_{n+1}}{\sum_{i=1}^{n} a_i} = a_{n+1}\Big(\sum_{i=1}^{n} a_i\Big)^{p}$$

This has a growth order of precisely $\mathfrak{a}(\Sigma\mathfrak{a})^p$, and the partial sums of $b_{n+1} - b_n$ are telescoping sums with a growth order of $\mathfrak{b}$, so we may conclude that $\Sigma(\mathfrak{a}(\Sigma\mathfrak{a})^p) = \mathfrak{b} = \mathbf{1}$ as claimed. □

The following is another very general summation formula that we will make extensive use of in Chapter 6 on closed chains, in order to construct growth orders with a prescribed partial sum ratio.

Proposition 50. If $\alpha = (a_n) \in \mathfrak{a}$ is a sequence tending to zero, and the sequence $\beta = e^{\Sigma\alpha}$ has growth order $\mathfrak{b}$, then $\Sigma(\mathfrak{a}\mathfrak{b}) = \mathfrak{b}$.

Proof. The equation $\beta = (b_n) = e^{\Sigma\alpha}$ means that

$$b_n = e^{\sum_{k=1}^{n} a_k}$$

Now, notice that

$$b_n - b_{n-1} = e^{\sum_{k=1}^{n} a_k} - e^{\sum_{k=1}^{n-1} a_k} = e^{\sum_{k=1}^{n} a_k}(1 - e^{-a_n})$$

Because $1 - e^{-h} = \Theta(h)$ as $h \longrightarrow 0$, and (a_n) is a sequence tending to zero, we have that the sequence $(1 - e^{-a_n})$ has the same growth order $\mathfrak{a}$ as (a_n), and therefore that the difference $b_n - b_{n-1}$ has growth order $\mathfrak{a}\mathfrak{b}$. However, the partial sums of the sequence $b_n - b_{n-1}$ are equal to b_n plus $O(1)$ by telescoping, and the $O(1)$ term can be neglected, since we already know that (b_n) is $\Omega(1)$ [4] (since e^{a_n} is bounded below by 1 for $a_n > 0$). Hence, we have that $\Sigma(\mathfrak{a}\mathfrak{b}) = \mathfrak{b}$ as claimed. □

[4] For those not familiar with this notation, for sequences of positive reals, a sequence x_n is $\Omega(y_n)$ iff y_n is $O(x_n)$.

As a brief aside, it is worth mentioning that one could think of the formulas introduced in Proposition 48, Proposition 49 and Proposition 50 as being analogous to the following integral identities from calculus:

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + C$$
$$\int f'(x)\cdot f(x)^p\,dx = \frac{f(x)^{p+1}}{p+1} + C$$
$$\int f'(x)\cdot e^{f(x)}\,dx = e^{f(x)} + C$$

Note that these are all essentially special cases of the chain rule. We will make this connection explicit later on in Proposition 61, when we prove a chain rule analogue for growth orders.

4.5. The convergence-divergence boundary

A natural question to ask while exploring convergent and divergent infinite series is the following: does there exist a growth order exhibiting the *slowest possible decay* for a sequence with divergent partial sums? That is, does there exist a minimal growth order whose partial sums diverge?

Proposition 51. For any growth order $\mathfrak{a}$ with $\Sigma\mathfrak{a} > \mathbf{1}$, there exists a growth order $\mathfrak{b} < \mathfrak{a}$ with $\Sigma\mathfrak{b} > \mathbf{1}$.

Proof. Choose an arbitrary $(a_n) \in \mathfrak{a}$. First, we consider the case in which a_n does not tend to zero. In this case, (a_n) must have some subsequence bounded below by a nonzero constant. We may construct a sequence (b_n) by replacing the terms of this subsequence with the respective elements of the harmonic sequence $(1/n)$. The partial sums of (b_n) diverge because of the divergence of the harmonic series, and $[(b_n)] < [(a_n)]$ because $a_n = b_n$ for terms not in this subsequence, and b_n/a_n tends to zero for terms belonging to this subsequence.

Now suppose that $(a_n) \in \mathfrak{a}$ tends to zero. Let (b_n) be the sequence defined by $b_1 = \sqrt{a_1}$ and

$$b_n = \sqrt{\sum_{i=1}^{n} a_i} - \sqrt{\sum_{i=1}^{n-1} a_i}$$

so that

$$\sum_{i=1}^{n} b_i = \sqrt{\sum_{i=1}^{n} a_i}$$

by telescoping. Since the partial sums of (a_n) tend to infinity, the partial sums of (b_n) also tend to infinity, because $x \mapsto \sqrt{x}$ is an unbounded strictly increasing function on $\mathbb{R}^+$. We also have that

$$\begin{aligned} b_n &= \left(\sqrt{\sum_{i=1}^{n-1} a_i}\right)\left(\sqrt{1+\frac{a_n}{\sum_{i=1}^{n-1} a_i}}-1\right) \\ &\sim \left(\sqrt{\sum_{i=1}^{n-1} a_i}\right)\cdot\frac{1}{2}\frac{a_n}{\sum_{i=1}^{n-1} a_i} \\ &= \frac{a_n}{2\sqrt{\sum_{i=1}^{n-1} a_i}} \end{aligned}$$

which has growth order $\mathfrak{a}/\sqrt{\Sigma\mathfrak{a}}$, which is strictly less than $\mathfrak{a}$, since $\Sigma\mathfrak{a} > \mathbf{1}$. Thus, we have found a growth order $\mathfrak{b}$ such that $\Sigma\mathfrak{b} > \mathbf{1}$ and $\mathfrak{b} < \mathfrak{a}$. □

This allows us to answer our question in the negative. There can be no "slowest diverging" infinite series, because for any growth order whose partial sums diverge, there exists a strictly lesser growth order whose partial sums also diverge. This means that for any growth order with divergent partial sums, we can, in fact, construct a *strictly decreasing infinite sequence* of growth orders starting with the given growth order, each of whose partial sums diverges. The following growth orders are familiar ones from calculus:

$$\cdots < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2\mathfrak{l}_3)^{-1} < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2)^{-1} < (\mathfrak{n}\mathfrak{l})^{-1} < \mathfrak{n}^{-1}$$

where the partial sums of the sequence $(\mathfrak{n}\mathfrak{l}\cdots\mathfrak{l}_m)^{-1}$ diverge with a growth order of $\mathfrak{l}_{m+1}$. (We will prove this fact later, in Proposition 91.) This provokes another question. We know that there is no *least* diverging growth order, but maybe some sequence of growth orders like the above "covers" all growth orders whose partial sums diverge. For instance, perhaps we can say that every growth order $\mathfrak{a}$ whose partial sums diverge falls above some growth order from the above list, so that no growth order diverges more slowly than *all* of the growth orders $(\mathfrak{n}\mathfrak{l}\cdots\mathfrak{l}_m)^{-1}$. In order-theoretic terms, we would say that the above sequence of growth orders is a base for the filter of growth orders with divergent partial sums. Is this the case?

Theorem 52. *Given any infinite descending sequence of growth orders*

$$\cdots < \mathfrak{a}_3 < \mathfrak{a}_2 < \mathfrak{a}_1$$

such that $\Sigma\mathfrak{a}_i > \mathbf{1}$ for all $i \in \mathbb{N}$, there always exists some growth order $\mathfrak{c}$ such that $\mathfrak{c} < \mathfrak{a}_i$ for all $i \in \mathbb{N}$ and yet $\Sigma\mathfrak{c} > \mathbf{1}$.

Proof. Let us start by choosing infinitely many sequences $(a_n^{(i)}) \in \mathfrak{a}_i$ from the given sequence of growth orders (using the Axiom of Choice). Because $\mathfrak{a}_{i+1} < \mathfrak{a}_i$ for each $i \in \mathbb{N}$, there exists some sequence of constants C_i (again, using the Axiom of Choice) such that

$$\frac{a_n^{(i+1)}}{a_n^{(i)}} \leq C_i$$

for all $n \in \mathbb{N}$, for each $i \in \mathbb{N}$. This inequality implies that

$$\frac{a_n^{(i+1)}}{a_n^{(j)}} \leq C_i C_{i-1} \cdots C_j$$

for all $n \in \mathbb{N}$, for all $j \leq i$. Denote the constant $C_i C_{i-1} \cdots C_j$ by $B_{i,j}$.

Let us now define another class of sequences $(b_n^{(i)})$ as follows: set $(b_n^{(1)}) = (a_n^{(1)})$, and

$$b_n^{(i+1)} = \frac{a_n^{(i+1)}}{\max(B_{i,1}, B_{i,2}, \cdots, B_{i,i})}$$

so that we have $b_n^{(i)} \leq b_n^{(j)}$ for all $n \in \mathbb{N}$ and $j \leq i$. In essence, we have normalized the sequences $(a_n^{(i)})$ so that each sequence $(b_n^{(i)})$ has the same growth order of $\mathfrak{a}_i$ for a fixed value of i, but $b_n^{(i)}$ is a decreasing function of i. In other words, if we arrange these sequences in a table:

$$\begin{array}{cccccc}
b_1^{(1)} & b_2^{(1)} & b_3^{(1)} & \cdots & b_n^{(1)} & \cdots \\
b_1^{(2)} & b_2^{(2)} & b_3^{(2)} & \cdots & b_n^{(2)} & \cdots \\
b_1^{(3)} & b_2^{(3)} & b_3^{(3)} & \cdots & b_n^{(3)} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
b_1^{(i)} & b_2^{(i)} & b_3^{(i)} & \cdots & b_n^{(i)} & \cdots
\end{array}$$

then the sequence along row i has growth order $\mathfrak{a}_i$, and the sequence down column n is monotone decreasing.

Since each sequence $b_n^{(i)}$ has divergent partial sums, for any given $M \in \mathbb{R}^+$, there exists some index m such that the sum of the first m terms of the sequence exceeds M. Therefore, we may define a more general function $\mathrm{ind}_M(i)$ as follows: let $\mathrm{ind}_M(i)$ be the smallest value of m such that the sum of the first m terms of the sequence $(b_n^{(i)})$ exceeds M. Notice that $\mathrm{ind}_M(i) \leq \mathrm{ind}_M(j)$ when $i \leq j$, since $b_n^{(i)} \geq b_n^{(j)}$ for $i \leq j$. Informally, for sequences on lower rows of the table, it takes their partial sums at least as long to reach large values.

Finally, let us define a new sequence (c_n), which will in some sense "diagonalize" over the above family of sequences. Define c_n piecewise as follows:

$$c_n = \begin{cases} b_n^{(1)} & \text{if } n \leq \mathrm{ind}_1(1) \\ b_n^{(i+1)} & \text{when } \mathrm{ind}_i(i) < n \leq \mathrm{ind}_{i+1}(i+1) \end{cases}$$

notice that these cases well-define c_n because $\mathrm{ind}_i(i)$ is a monotone increasing sequence of i, and therefore every $n \geq \mathrm{ind}_i(i)$ falls in the interval of integers $(\mathrm{ind}_i(i), \mathrm{ind}_{i+1}(i+1)]$ for exactly one value of $i \in \mathbb{N}$. (Some of these intervals are empty, namely the ones where $\mathrm{ind}_i(i) = \mathrm{ind}_{i+1}(i+1)$.) At this point we just need to show that (c_n) satisfies the desired properties of having growth order less than each $\mathfrak{a}_i$, and having divergent partial sums.

First of all, notice that for all $n > \mathrm{ind}_i(i)$, we have that $c_n \leq b_n^{(i)}$, since for these values of n we will have $c_n = b_n^{(j)}$ for some values $j > i$, and we have already shown that $b_n^{(j)} \leq b_n^{(i)}$ for $j \geq i$. Thus, since c_n is bounded above by $b_n^{(i)}$ for all but finitely many values of n. This means that, if $\mathfrak{c} = [(c_n)]$, we have $\mathfrak{c} \leq \mathfrak{a}_i$ for each i, and hence $\mathfrak{c} \leq \mathfrak{a}_{i+1} < \mathfrak{a}_i$ and $\mathfrak{c} < \mathfrak{a}_i$ for each $i \in \mathbb{N}$, as claimed.

Finally, consider the sum of the first $\mathrm{ind}_i(i)$ values of c_n. For each $n \leq \mathrm{ind}_i(i)$, we have that $c_n = b_n^{(j)}$ for some $j < i$, meaning that $c_n \geq b_n^{(i)}$ for each $n \leq \mathrm{ind}_i(i)$. But, by the definition of $\mathrm{ind}_i(i)$, we have that the sum $b_1^{(i)} + \cdots + b_{\mathrm{ind}_i(i)}^{(i)} \geq i$, meaning that we also have $c_1 + \cdots + c_{\mathrm{ind}_i(i)} \geq i$. Thus, the partial sums of (c_n) are unbounded, and $\Sigma\mathfrak{c} > \mathbf{1}$, as desired. □

Again, our question is answered negatively. Given any descending sequence of growth orders with divergent partial sums, there exists a growth orders with partial sums that diverge even more slowly.

What does this look like in terms of the familiar sequence of growth orders

$$\cdots < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2\mathfrak{l}_3)^{-1} < (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2)^{-1} < (\mathfrak{n}\mathfrak{l})^{-1} < \mathfrak{n}^{-1}$$

mentioned earlier? Well, let us repeat the construction for this family of growth orders. Our family of sequences $(a_n^{(i)})$ can be defined as follows:

$$\begin{aligned}
a_n^{(1)} &= n^{-1} \\
a_n^{(2)} &= n^{-1}(1 + \log_2 n)^{-1} \\
a_n^{(3)} &= n^{-1}(1 + \log_2 n)^{-1}\big(1 + \log_2(1 + \log_2 n)\big)^{-1} \\
&\cdots
\end{aligned}$$

For these sequences, we already have $a_n^{(j)} \leq a_n^{(i)}$ for all $i \leq j$, so we do not even need to bother with normalizing our sequences. The partial sums of $a_n^{(i)}$ have growth order $\mathfrak{l}_i$, meaning that $\mathrm{ind}_i(i)$ will look something like ${}^{i}2$, where the left superscript denotes tetration - that is, a power tower consisting of i many 2s. Therefore, if we let $\mathrm{slog}_2(n)$ denote the smallest natural number i such that $\mathrm{ind}_i(i) < i$, then our diagonalizing sequence (c_n) could be given by

$$c_n = n^{-1} \cdot (1 + \log_2 n)^{-1} \cdot \big(1 + \log_2(1 + \log_2 n)\big)^{-1} \cdot \cdots \cdot \Big(\overbrace{1 + \log_2\big(\cdots(1 + \log_2 n)\cdots\big)}^{\mathrm{slog}_2(n)\text{ nested logarithms}}\Big)^{-1}$$

This sequence decays faster than each growth order $(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$, yet its partial sums still diverge. (Note: the name slog_2 is chosen as a reference to the so-called "super-logarithm", which is sometimes defined as an analogue of the logarithm for tetration rather than exponentiation.)

5. Composition and inverses

5.1. Definition of the composite

Given a sequence (a_n), we might want to examine the ways in which its growth order can change when it is reindexed. For instance, we may want to consider subsequences like (a_{2n}), which has the same growth order as (a_n) given moderate growth as we proved in Section 2.3, or (a_{n^2}). These subsequences *accelerate* the growth or decay of the sequence (a_n), but we might also consider subsequences like $(a_{\lfloor\sqrt{n}\rfloor})$ that "slow down" the original sequence.

In general, we may want to consider (a_{b_n}) for an arbitrary indexing sequence (b_n). However, this only makes sense when b_n is a sequence of natural numbers, since a_n is only defined for $n \in \mathbb{N}$. Hence, if we want to define the composite of two sequences $\alpha \circ \beta = (a_{b_n})$, we need (b_n) to consist of natural numbers.

Definition 53. Given sequences $\alpha \in \mathcal{S}(\mathbb{R}^+)$ and $\beta \in \mathcal{S}(\mathbb{N})$, define their **composite**, denoted by $\alpha \circ \beta$, to be the sequence (a_{b_n}), and let β be called the **indexing sequence**.

However, even if (b_n) is not a sequence of natural numbers, there may still exist a sequence of natural numbers *of the same growth order* which could be used as a sequence of indices in place of (b_n). In fact, the following proposition proves that such a sequence exists whenever $\mathfrak{b} \geq \mathbf{1}$.

Proposition 54. For every sequence $\alpha = (a_n) \in \mathcal{S}(\mathbb{R}^+)$ with $[\alpha] \geq \mathbf{1}$, there exists a sequence of natural numbers $\beta \in \mathcal{S}(\mathbb{N})$ such that $[\alpha] = [\beta]$.

Proof. Let $[\alpha] \geq \mathbf{1}$. Then we shall show that the sequence $\beta = (b_n) \in \mathcal{S}(\mathbb{N})$ defined by $b_n = \lceil a_n \rceil$ has the same growth order as α. Since $\lceil x \rceil - x \in [0, 1)$ for all $x \in \mathbb{R}^+$, it follows that $b_n - a_n \in [0, 1)$ and therefore

$$a_n \leq b_n \leq a_n + 1$$

for all $n \in \mathbb{N}$. Since $[\alpha] \geq \mathbf{1}$, we have that $a_n \geq C$ for some $C \in \mathbb{R}^+$, meaning that $a_n + 1 \leq (1 + C^{-1})a_n$, and therefore

$$a_n \leq b_n \leq (1 + C^{-1})a_n$$

for all $n \in \mathbb{N}$, proving that $[\alpha] = [\beta]$ as claimed. □

Now that we can define the composite of two sequences α, β with $[\beta] \geq \mathbf{1}$, we'd like to define it on growth orders as well. The most natural definition would be to let $\mathfrak{a} \circ \mathfrak{b} = [\alpha] \circ [\beta] = [\alpha \circ \beta]$.

However, we must show that this operation is well-defined for the class of growth orders that we are most concerned with - namely, the moderate ones.

Proposition 55. Let $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ and $\beta, \beta' \in \mathcal{S}(\mathbb{N})$. If $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, and α, α' exhibit moderate growth, then $[\alpha \circ \beta] = [\alpha' \circ \beta']$.

Proof. It suffices to show the following two facts:

(1) If $\alpha, \alpha' \in \mathcal{S}(\mathbb{R}^+)$ are moderate with $\alpha \sim \alpha'$ and $\beta \in \mathcal{S}(\mathbb{N})$, then $\alpha \circ \beta \sim \alpha' \circ \beta$.

(2) If $\alpha \in \mathcal{S}(\mathbb{R}^+)$ is moderate and $\beta, \beta' \in \mathcal{S}(\mathbb{N})$ with $\beta \sim \beta'$, then $\alpha \circ \beta \sim \alpha \circ \beta'$.

In other words, we are showing that the growth order of $\alpha \circ \beta$ depends only on the respective growth orders of α and β.

The former claim (1) can be shown quickly: because $\alpha \sim \alpha'$, we have that there exist constants $C_1, C_2 > 0$ such that

$$C_1 a_n \leq a'_n \leq C_2 a_n$$

for all $n \in \mathbb{N}$. Because this holds for all natural numbers n, we may replace n with b_n, obtaining the inequality

$$C_1 a_{b_n} \leq a'_{b_n} \leq C_2 a_{b_n}$$

which means that $\alpha \circ \beta \sim \alpha' \circ \beta$ by definition.

The proof of the latter claim is a little more cumbersome. Let α, β, β' be given as in (2), and let $C_1, C_2 > 0$ be constants such that

$$C_1 b_n \leq b'_n \leq C_2 b_n$$

for all $n \in \mathbb{N}$. This implies the following weaker inequality, since $\lceil x \rceil \geq x$ and $\lceil x^{-1} \rceil^{-1} \leq x$ for all $x > 0$:

$$\lceil C_1^{-1} \rceil^{-1} b_n \leq b'_n \leq \lceil C_2 \rceil b_n$$

Therefore, we have natural numbers $K_1 = \lceil C_1^{-1} \rceil$ and $K_2 = \lceil C_2 \rceil$ such that

$$K_1^{-1} b_n \leq b'_n \leq K_2 b_n$$

Now, because b'_n is an integer, and $\lceil K_1^{-1} b_n \rceil$ is the smallest integer greater than or equal to $K_1^{-1} b_n$, we have that

$$\lceil K_1^{-1} b_n \rceil \leq b'_n \leq K_2 b_n$$

Further, notice that $b_n \leq K_1 \lceil K_1^{-1} b_n \rceil$, since $\lceil K^{-1} b_n \rceil \geq K^{-1} b_n$ by the definition of the ceiling. Thus, we may loosen the upper bound by replacing b_n with $K_1 \lceil K_1^{-1} b_n \rceil$:

$$\lceil K_1^{-1} b_n \rceil \leq b'_n \leq K_2 K_1 \lceil K_1^{-1} b_n \rceil$$

Now we shall make use of the moderateness property of α. First, we may let $D_1 > 0$ be a constant such that

$$a_m \leq D_1 a_n$$

for all $n \le m \le K_2K_1n$. Secondly, we may let $D_2 > 0$ be a constant such that $a_n \le D_2a_m$ for all $n \le m \le (K_1+1)n$. We shall use these definitions to procure a chain of inequalities culminating in the inequality $a_{b'_n} \le D_1D_2^2a_{b_n}$. First of all, we have that

$$a_{b'_n} \le D_1a_{\lceil K_1^{-1}b_n\rceil}$$

by the definition of D_1, and because $\lceil K_1^{-1}b_n\rceil \le b'_n \le K_2K_1\lceil K_1^{-1}b_n\rceil$ as proven earlier. Next, we have the inequality

$$D_1a_{\lceil K_1^{-1}b_n\rceil} \le D_1D_2a_{K_1\lceil K_1^{-1}b_n\rceil}$$

by the definition of D_2, and because $\lceil K_1^{-1}b_n\rceil \le K_1\lceil K_1^{-1}b_n\rceil \le (K_1+1)\lceil K_1^{-1}b_n\rceil$. Finally, we have the inequality

$$D_1D_2a_{K_1\lceil K_1^{-1}b_n\rceil} \le D_1D_2^2a_{b_n}$$

which follows from the definition of D_2 and the very weak inequality $b_n \le K_1\lceil K_1^{-1}b_n\rceil \le (K_1+1)b_n$. Thus, we have established the following chain of 3 inequalities:

$$a_{b'_n} \le D_1a_{\lceil K_1^{-1}b_n\rceil} \le D_1D_2a_{K_1\lceil K_1^{-1}b_n\rceil} \le D_1D_2^2a_{b_n}$$

which, at least, tells us that $a_{b'_n} \le D_1D_2^2a_{b_n}$, or that $[\alpha\circ\beta'] \le [\alpha\circ\beta]$. Because β,β' were completely arbitrary, we have by symmetry that $[\alpha\circ\beta] \le [\alpha\circ\beta']$ as well, meaning that $\alpha\circ\beta \sim \alpha\circ\beta'$. This completes the proof of (2). □

This proves that the composition operation can be well-defined on *growth orders* of sequences, not just individual sequences. In particular, we may define the composition of growth orders $\mathfrak{a}\circ\mathfrak{b}$ whenever $\mathfrak{a}$ is moderate and $\mathfrak{b} \ge \mathbf{1}$, by choosing an arbitrary sequence $\alpha \in \mathfrak{a}$ and an indexing sequence of natural numbers $\beta \in \mathfrak{b}$ (whose existence is guaranteed by Proposition 54) and considering the growth order $[\alpha\circ\beta]$. This is well-defined because this equivalence class is independent of the choice of α and β, as we have just proven.

Note that not only is $\mathfrak{a}\circ\mathfrak{b}$ defined when $\mathfrak{a}$ is moderate (as proven above), but moderateness is *precisely* what is needed for $\mathfrak{a}\circ\mathfrak{b}$ to be well-defined. If $\mathfrak{a}$ is *not* moderate, then we may show that there *always* exist sequences $\alpha \in \mathfrak{a}$ and $\beta,\beta' \in \mathfrak{b} = \mathfrak{n}$ such that $[\alpha\circ\beta] \ne [\alpha\circ\beta']$, so that $\mathfrak{a}\circ\mathfrak{n}$ is not even well-defined! This is because if $\mathfrak{a}$ is immoderate, then (by the Axiom of Choice) there exists a sequence of indices (m_n) such that $n \le m_n \le 2n$ for each $n \in \mathbb{N}$ but such that (a_{m_n}) is unbounded, by the contrapositive of Proposition 7. Hence, if we let $\beta = (n)$ and $\beta' = (m_n)$, we have that $\beta,\beta' \in \mathfrak{n}$ but $[\alpha\circ\beta] \ne [\alpha\circ\beta']$. Hence, we can alternatively think of "moderateness" as the property of being "well-behaved with respect to composition".

Definition 56. Given growth orders $\mathfrak{a},\mathfrak{b}$ with $\mathfrak{a}$ moderate and $\mathfrak{b} \ge 1$, define their **composite** $\mathfrak{a}\circ\mathfrak{b}$ as the equivalence class $[\alpha\circ\beta]$, where $\alpha \in \mathfrak{a}$ and $\beta \in \mathfrak{b}\cap\mathcal{S}(\mathbb{N})$ are arbitrary.

If we consider growth orders with properties that make them amenable to both left- and right-composition, we can form subsets of $\mathcal{S}(\mathbb{R}^+)$ that are closed under composition. Such subsets may carry the structure of a *monoid*, since the binary operation of composition is associative (as we will see in Proposition 60).

Proposition 57. If $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)$ consists of moderate growth orders ≥ 1 and is closed under composition, then it is a *semigroup* [a] under composition. If it contains $\mathfrak{n}$, then it is a *monoid* [b] under composition, with identity element $\mathfrak{n}$.

[a] A semigroup is a set endowed with an associative binary operation.
[b] A monoid is a semigroup for which the binary operation has an identity element.

Since the properties of moderateness and monotonicity have proven useful to us so far, it is worth showing that composition of growth orders preserves these properties.

Proposition 58. If $\mathfrak{a}$ and $\mathfrak{b} \geq 1$ are moderate, then $\mathfrak{a} \circ \mathfrak{b}$ is moderate.

Proof. Let $(a_n) \in \mathfrak{a}$ be arbitrarily chosen, and let $(b_n) \in \mathfrak{b} \cap \mathcal{S}(\mathbb{N})$ be arbitrary. By the moderateness of $\mathfrak{b}$, we immediately have that there exist constants $C_1, C_2 > 0$ such that

$$C_1 b_n \leq b_m \leq C_2 b_n$$

for $n \leq m \leq 2n$. Then if we let $K = \max(\lceil C_2 \rceil, \lceil C_1^{-1} \rceil)$, we have that K is an integer such that

$$\frac{1}{K} b_n \leq b_m \leq K b_n$$

or equivalently

$$b_n \leq K b_m \leq K^2 b_n$$

for all $n \leq m \leq 2n$. Next, by the moderateness of $\mathfrak{a}$, there exist constants $C_3, C_4 > 0$ such that

$$C_3 a_n \leq a_m \leq C_4 a_n$$

whenever $n \leq m \leq K^2 n$. By the previously derived inequalities for the sequence (b_n), this implies that

$$a_{Kb_m} \leq C_4 a_{b_n} \leq C_3^{-1} C_4 a_{Kb_n}$$

and additionally

$$a_{Kb_m} \geq C_3 a_{b_n} \geq C_3 C_4^{-1} a_{Kb_n}$$

whenever $n \leq m \leq 2n$, so that we have

$$C_3 C_4^{-1} a_{Kb_n} \leq a_{Kb_m} \leq C_3^{-1} C_4 a_{Kb_n}$$

whenever $n \leq m \leq 2n$. By Proposition 7, this is sufficient to show that the sequence (a_{Kb_n}) is moderate, and since composition is well-defined on growth orders, that $\mathfrak{a} \circ \mathfrak{b}$ is moderate.

□

Proving that composition preserves monotonicity is much more straightforward:

Proposition 59. If $\mathfrak{a}$ is monotone and $\mathfrak{b} \geq 1$ are monotone, then $\mathfrak{a} \circ \mathfrak{b}$ is monotone.

Proof. Let $(a_n) \in \mathfrak{a}$ be monotone and $(b_n) \in \mathfrak{b}$ be a monotone increasing sequence of positive integers. If (a_n) is monotone increasing, then $i \leq j$ implies $b_i \leq b_j$ and $a_{b_i} \leq a_{b_j}$, so that (a_{b_n}) is also monotone increasing. If (a_n) is monotone decreasing, then $i \leq j$ implies $b_i \leq b_j$ and $a_{b_i} \geq a_{b_j}$, so that (a_{b_n}) is also monotone decreasing. In either case, (a_{b_n}) is monotone, so we have that $\mathfrak{a} \circ \mathfrak{b}$ is monotone. □

5.2. Arithmetic and inequalities

Here are some elementary properties of composition, and its interactions with other operations on growth orders:

Proposition 60. The following equalities hold for growth orders $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ whenever the stated composites are defined:

- $(\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{c} = \mathfrak{a} \circ (\mathfrak{b} \circ \mathfrak{c})$
- $(\mathfrak{a} + \mathfrak{b}) \circ \mathfrak{c} = \mathfrak{a} \circ \mathfrak{b} + \mathfrak{a} \circ \mathfrak{c}$
- $\mathfrak{a}\mathfrak{b} \circ \mathfrak{c} = (\mathfrak{a} \circ \mathfrak{c})(\mathfrak{b} \circ \mathfrak{c})$
- $\mathfrak{a}^p = \mathfrak{n}^p \circ \mathfrak{a}$

Proof. Let us consider sequences of positive reals as functions $\mathbb{N} \to \mathbb{R}^+$, and select some f, g, h such that $(f(n)) \in \mathfrak{a}$, $(g(n)) \in \mathfrak{b}$, and $(h(n)) \in \mathfrak{c}$ with all values of g, h being natural numbers. We have that $(\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{c}$ is the growth order of $((f \circ g) \circ h)(n)$, whereas $\mathfrak{a} \circ (\mathfrak{b} \circ \mathfrak{c})$ is the growth order of $(f \circ (g \circ h))(n)$, but we have that

$$((f \circ g) \circ h)(n) = (f \circ (g \circ h))(n)$$

because function composition is associative, showing that the two desired growth orders are equal. We may use the same strategy for the second and third equalities, using the facts that

$$(f + g) \circ h = (f \circ h) + (g \circ h)$$

$$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

For the final equality, if $f(n) = n^p$ for some $p \in \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{N}$, then we have that

$$f(g(n)) = g(n)^p$$

which demonstrates the fourth equality. □

Earlier in Proposition 4.4 we proved several "cookbook-style" formulas for partial sums that were reminiscent of integral identities from calculus, all essentially special cases of the chain rule. The chain rule relates function composition to differentiation, so without an analogue of composition for growth orders, we were not prepared to formalize this analogy rigorously. However, we now have the adequate machinery to do so.

Theorem 61. ***(Chain rule analogue.)*** *If $\mathfrak{a}, \mathfrak{b}$ are moderate and $\mathfrak{b} \geq \mathbf{1}$, then*

$$(\Sigma\mathfrak{a}) \circ (\Sigma\mathfrak{b}) = \Sigma\big(\mathfrak{b} \cdot (\mathfrak{a} \circ \Sigma\mathfrak{b})\big)$$

Proof. Let $\alpha = (a_n) \in \mathfrak{a}$ and $\beta = (b_n) \in \mathfrak{b}$ with (b_n) a sequence of natural numbers, and define $(A_n) = \Sigma\alpha$ and $(B_n) = \Sigma\beta$. (We will also adopt the convention that $B_0 = 0$.) Because (b_n) and hence (B_n) is moderate, we have that $B_j \leq KB_{j-1}$ for all $j > 1$, for some integer $K > 0$. Further, because (a_n) is moderate, there exist constants $C_1, C_2 > 0$ such that $C_1 a_n \leq a_m \leq C_2 a_n$ for all $n \leq m \leq Kn$.

These bounds imply that

$$\begin{aligned}
\sum_{k=1}^{B_n} a_k &= \sum_{j=1}^{n} \sum_{k=B_{j-1}+1}^{B_j} a_k \\
&\leq \sum_{j=1}^{n} \sum_{k=B_{j-1}+1}^{B_j} C_2 a_{B_{j-1}+1} \\
&= \sum_{j=1}^{n} C_2 (B_j - B_{j-1}) a_{B_{j-1}+1} \\
&= \sum_{j=1}^{n} C_2 b_j a_{B_{j-1}+1}
\end{aligned}$$

and further that

$$\begin{aligned}
\sum_{k=1}^{B_n} a_k &= \sum_{j=1}^{n} \sum_{k=B_{j-1}+1}^{B_j} a_k \\
&\geq \sum_{j=1}^{n} \sum_{k=B_{j-1}+1}^{B_j} C_1 a_{B_{j-1}+1} \\
&= \sum_{j=1}^{n} C_1 (B_j - B_{j-1}) a_{B_{j-1}+1} \\
&= \sum_{j=1}^{n} C_1 b_j a_{B_{j-1}+1}
\end{aligned}$$

Together, these bounds give

$$\sum_{j=1}^{n} C_1 b_j a_{B_{j-1}+1} \leq \sum_{k=1}^{B_n} a_k \leq \sum_{j=1}^{n} C_2 b_j a_{B_{j-1}+1}$$

The middle sum has growth order $(\Sigma\mathfrak{a}) \circ (\Sigma\mathfrak{b})$, and each of the outer sums has growth order $\Sigma(\mathfrak{b} \cdot \mathfrak{a} \circ \Sigma\mathfrak{b})$. This proves the desired claim. $\square$

Proposition 62. The following inequalities hold for growth orders $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ whenever the stated composites are defined:

$$\begin{aligned} \mathfrak{a} \leq \mathfrak{b} &\implies \mathfrak{a} \circ \mathfrak{c} \leq \mathfrak{b} \circ \mathfrak{c} \\ \mathfrak{a} \geq 1 \text{ monotone}, \mathfrak{b} \leq \mathfrak{c} &\implies \mathfrak{a} \circ \mathfrak{b} \leq \mathfrak{a} \circ \mathfrak{c} \\ \mathfrak{a} \leq 1 \text{ monotone}, \mathfrak{b} \leq \mathfrak{c} &\implies \mathfrak{a} \circ \mathfrak{b} \geq \mathfrak{a} \circ \mathfrak{c} \end{aligned}$$

Proof. For the first claim, suppose $\mathfrak{a} \leq \mathfrak{b}$ and let $(c_n) \in \mathfrak{c}$ be a sequence of positive integers. If $(a_n) \in \mathfrak{a}$ and $(b_n) \in \mathfrak{b}$ such that a_n/b_n is bounded above, then we have that a_{c_n}/b_{c_n} is bounded above by the same upper bound. The sequences a_{c_n} and b_{c_n} have growth orders $\mathfrak{a} \circ \mathfrak{c}$ and $\mathfrak{b} \circ \mathfrak{c}$ respectively, so we have that $\mathfrak{a} \circ \mathfrak{c} \leq \mathfrak{b} \circ \mathfrak{c}$.

For the second claim, suppose that $(a_n) \in \mathfrak{a}$ is monotone increasing and $(b_n) \in \mathfrak{b}$, $(c_n) \in \mathfrak{c}$. If $\mathfrak{b} \leq \mathfrak{c}$, then b_n/c_n is bounded above by some positive constant C. If we define another sequence $b'_n = b_n/C$, then we have that $b'_n \leq c_n$ for all $n \in \mathbb{N}$ and $(b'_n) \in \mathfrak{b}$. Since (a_n) is monotone increasing we have that $a_{b'_n} \leq a_{c_n}$ and therefore $\mathfrak{a} \circ \mathfrak{b} \leq \mathfrak{a} \circ \mathfrak{c}$. The argument is almost identical for the third claim, except that (a_n) will be monotone decreasing. $\square$

Note that the restriction to monotone growth orders $\mathfrak{a}$ in the latter two claims is essential, for neither implication is necessarily true for non-monotone growth orders $\mathfrak{a}$. Consider, for instance, the sequence (a_n) defined by

$$a_n = n^{1+\sin\log\log n}$$

If $\mathfrak{a}$ is the growth order of this sequence, then we proved in section 4.2, page 31 that $\mathfrak{a}/\mathfrak{n}$ is moderate, hence $\mathfrak{a}$ is moderate. Consider the indexing sequences $(b_n), (c_n)$ given by $b_n = n$ and $c_n = \lfloor n^p \rfloor$, where $p = e^\pi$. We have that $|c_n - n^p| \leq 1$, and therefore

$$|\log\log c_n - \log\log n| = \pi + o(1)$$

which means that, since the sine function is continuous,

$$\sin\log\log c_n = -\sin\log\log b_n + o(1)$$

hence

$$\frac{a_{b_n}}{a_{c_n}} = \frac{n^{1+\sin(\log\log n)}}{n^{1+\sin(\log\log n \pm \pi + o(1))}} = n^{2\sin\log\log n} \cdot n^{o(1)}$$

This sequence is not comparable to $\mathbf{1}$, since it has subsequences tending to 0 and to ∞, as can be seen by showing that $\sin\log\log n + o(1)$ is $> 1/2$ for infinitely many $n \in \mathbb{N}$, and similarly $< -1/2$ for infinitely many $n \in \mathbb{N}$. Hence, we have that $\mathfrak{a} \circ \mathfrak{b} \perp \mathfrak{a} \circ \mathfrak{c}$, showing that $\mathfrak{a} \geq 1$ and $\mathfrak{b} \leq \mathfrak{c}$ do not necessarily imply that $\mathfrak{a} \circ \mathfrak{b} \leq \mathfrak{a} \circ \mathfrak{c}$.

5.3. Absorption

Sometimes reindexing a certain sequence by another sequence does not change its growth order at all. For instance, if we reindex a sequence of growth order $\mathfrak{l}$ by a subsequence with indices of growth order $\mathfrak{n}^2$, the growth order of the resulting sequence will just be $\mathfrak{l} = \mathfrak{l} \circ \mathfrak{n}^2$ again. Reindexing may have "accelerated" the sequence somewhat, but not significantly enough to affect its growth order, i.e. not by more than a constant factor. In this section, we explore when reindexing leaves a sequence's growth order unchanged.

Definition 63. Given growth orders $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{a} \circ \mathfrak{b}$ is defined, we say that $\mathfrak{a}$ **absorbs** $\mathfrak{b}$ if $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{a}$.

The following proposition shows that absorption is a transitive relation on composable growth orders.

Proposition 64. If $\mathfrak{a}$ absorbs $\mathfrak{b}$ and $\mathfrak{b}$ absorbs $\mathfrak{c}$, then $\mathfrak{a}$ absorbs $\mathfrak{c}$.

Proof. Given that the compositions $\mathfrak{a} \circ \mathfrak{b}$ and $\mathfrak{b} \circ \mathfrak{c}$ are defined, with $\mathfrak{a}$ absorbing $\mathfrak{b}$ and $\mathfrak{b}$ absorbing $\mathfrak{c}$, we have that

$$\mathfrak{a} \circ \mathfrak{c} = (\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{c} = \mathfrak{a} \circ (\mathfrak{b} \circ \mathfrak{c}) = \mathfrak{a} \circ \mathfrak{b} = \mathfrak{a}$$

and therefore $\mathfrak{a} \circ \mathfrak{c} = \mathfrak{a}$ and $\mathfrak{a}$ absorbs $\mathfrak{c}$. □

Back in Proposition 43 we proved a "squeezing property" for partial summation, and we can also prove an analogous "squeezing property" for absorption. We showed that if $\mathrm{P}\mathfrak{b}_1$ and $\mathrm{P}\mathfrak{b}_2$ are equal, and $\mathfrak{a}$ is trapped between $\mathfrak{b}_1$ and $\mathfrak{b}_2$ having a monotone quotient with both of them, then $\mathrm{P}\mathfrak{a}$ is equal to the same ratio. The analogous property of absorption is that if $\mathfrak{b}$ absorbs $\mathfrak{c}$, and $\mathfrak{a}$ is trapped between $\mathbf{1}$ and $\mathfrak{b}$ with monotone quotients, then it must also absorb $\mathfrak{c}$.

The following proposition will serve as a lemma to help us prove the desired property:

Proposition 65. If $\mathfrak{a}, \mathfrak{b}$ are moderate growth orders with $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{a}/\mathfrak{b}$ monotone, and if $\mathfrak{c}$ is some growth order $\geq \mathfrak{n}$, then

$$\frac{\mathfrak{a} \circ \mathfrak{c}}{\mathfrak{a}} \leq \frac{\mathfrak{b} \circ \mathfrak{c}}{\mathfrak{b}}$$

Proof. Since $\mathfrak{a}/\mathfrak{b} \leq \mathbf{1}$ is monotone and $\mathfrak{c} \geq \mathfrak{n}$, we have by Proposition 62 that

$$\frac{\mathfrak{a}}{\mathfrak{b}} \circ \mathfrak{c} \leq \frac{\mathfrak{a}}{\mathfrak{b}} \circ \mathfrak{n} = \frac{\mathfrak{a}}{\mathfrak{b}}$$

or equivalently

$$\frac{\mathfrak{a} \circ \mathfrak{c}}{\mathfrak{b} \circ \mathfrak{c}} \leq \frac{\mathfrak{a}}{\mathfrak{b}}$$

Multiplying both sides by $(\mathfrak{b} \circ \mathfrak{c})/\mathfrak{a}$ yields the desired result:

$$\frac{\mathfrak{a} \circ \mathfrak{c}}{\mathfrak{a}} \leq \frac{\mathfrak{b} \circ \mathfrak{c}}{\mathfrak{b}}$$

□

Finally, here is the "squeezing property" we wanted to prove:

Proposition 66. Let $\mathfrak{a}, \mathfrak{b}$ be moderate and monotone growth orders such that $\mathfrak{b}/\mathfrak{a}$ is monotone and $\mathbf{1} \leq \mathfrak{a} \leq \mathfrak{b}$. If $\mathfrak{c} \geq \mathfrak{n}$ and $\mathfrak{b}$ absorbs $\mathfrak{c}$, then $\mathfrak{a}$ also absorbs $\mathfrak{c}$.

Proof. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be as hypothesized. Then we have by the Proposition 65 that

$$\frac{\mathfrak{a} \circ \mathfrak{c}}{\mathfrak{a}} \leq \frac{\mathfrak{b} \circ \mathfrak{c}}{\mathfrak{b}}$$

Since $\mathfrak{a} \geq \mathbf{1}$ is monotone and $\mathfrak{c} \geq \mathfrak{n}$, we have that $(\mathfrak{a} \circ \mathfrak{c}) \geq \mathfrak{a}$, or $(\mathfrak{a} \circ \mathfrak{c})/\mathfrak{a} \geq \mathbf{1}$. However, since $\mathfrak{b}$ absorbs $\mathfrak{c}$, we have that $\mathfrak{b} \circ \mathfrak{c} = \mathfrak{b}$, or $(\mathfrak{b} \circ \mathfrak{c})/\mathfrak{b} = \mathbf{1}$. Hence, we have that

$$\mathbf{1} \leq \frac{\mathfrak{a} \circ \mathfrak{c}}{\mathfrak{a}} \leq \mathbf{1}$$

so that it must be the case that $(\mathfrak{a} \circ \mathfrak{c})/\mathfrak{a} = \mathbf{1}$ or $\mathfrak{a} \circ \mathfrak{c} = \mathfrak{a}$, meaning that $\mathfrak{a}$ absorbs $\mathfrak{c}$ as claimed. □

The following proposition shows that it is easy to take partial sums of growth orders that are "very absorbant":

Proposition 67. If $\mathfrak{a}$ is monotone and absorbs some $\mathfrak{b} < \min(\mathfrak{n}, \mathfrak{a}\mathfrak{n})$, then $\Sigma\mathfrak{a} = \mathfrak{n}\mathfrak{a}$.

Proof. Let $(a_n) \in \mathfrak{a}$ be monotone, and choose $(b_n) \in \mathfrak{b}$ to be a sequence of integers such that $b_n \leq n/2$ for all $n \in \mathbb{N}$ with $n > 1$. Note that (a_n) must also be moderate, in order for the composition $\mathfrak{a} \circ \mathfrak{b}$ to be defined at all.

First suppose (a_n) is monotone increasing. Then we have that

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} a_n = na_n$$

Thus, $\Sigma\mathfrak{a} \leq \mathfrak{n}\mathfrak{a}$. We also know that $\Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$ by Proposition 37, since $\mathfrak{a}$ is moderate. Hence, we have $\mathfrak{n}\mathfrak{a} \leq \Sigma\mathfrak{a} \leq \mathfrak{n}\mathfrak{a}$, and therefore $\Sigma\mathfrak{a} = \mathfrak{n}\mathfrak{a}$.

Now suppose that (a_n) is monotone decreasing. This time we have that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{b_n} a_i + \sum_{i=b_n+1}^{n} a_i \leq a_1 b_n + (n - b_n) a_{b_n} \sim n a_n$$

because $(n - b_n)a_{b_n} \sim (n - b_n)a_n$ (since $\mathfrak{a}$ absorbs $\mathfrak{b}$), and $(n - b_n)a_n \sim na_n$ (since $b_n \leq n/2$). This means $\Sigma\mathfrak{a} \leq \mathfrak{n}\mathfrak{a}$. But again, we already know $\Sigma\mathfrak{a} \geq \mathfrak{n}\mathfrak{a}$ by moderateness. Thus, we again have $\mathfrak{n}\mathfrak{a} \leq \Sigma\mathfrak{a} \leq \mathfrak{n}\mathfrak{a}$ and therefore $\Sigma\mathfrak{a} = \mathfrak{n}\mathfrak{a}$ as claimed. □

5.4. Inverses and cancellation

Definition 68. Given growth orders $\mathfrak{a}$ and $\mathfrak{b}$, if $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{n}$, then we say that $\mathfrak{b}$ is a **right inverse** of $\mathfrak{a}$, and that $\mathfrak{a}$ is a **left inverse** of $\mathfrak{b}$.

A right inverse of a growth order can be thought of as a way of reindexing sequences of that growth order in such a way that they exhibit linear growth. A left inverse, on the other hand, can be thought of as a sequence whose subsequences along a given reindexing exhibit linear growth.

We remarked earlier that subsets $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)$ consisting of moderate growth orders $\geq \mathbf{1}$ carry a *monoid structure* if they are closed under composition. From the monoid structure alone, we can deduce a couple basic properties of inverses when they exist, using standard arguments from abstract algebra. For instance:

Proposition 69. If a moderate growth order $\mathfrak{a} \geq \mathbf{1}$ has both a left-inverse $\mathfrak{b}$ and a right-inverse $\mathfrak{c}$, then $\mathfrak{b} = \mathfrak{c}$.

Proof. Suppose that $\mathfrak{b} \circ \mathfrak{a} = \mathfrak{a} \circ \mathfrak{c} = \mathfrak{n}$. Then we have that $\mathfrak{b} \circ (\mathfrak{a} \circ \mathfrak{c}) = \mathfrak{b} \circ \mathfrak{n}$. The LHS of this equality is equal to $\mathfrak{c}$, since $\mathfrak{b} \circ \mathfrak{a} = \mathfrak{n}$ and $\mathfrak{n} \circ \mathfrak{c} = \mathfrak{c}$, making use of associativity. The RHS is equal to $\mathfrak{b} \circ \mathfrak{n} = \mathfrak{b}$. Thus, we have $\mathfrak{c} = \mathfrak{b}$ as claimed. □

Proposition 70. If $\mathfrak{a} > \mathbf{1}$ is moderate, then it has a right-inverse.

Proof. Let $(a_n) \in \mathfrak{a}$ and let (b_n) be a sequence defined by letting b_n be the least integer m such that $a_m \geq n$. Such an integer always exists because $\mathfrak{a} > \mathbf{1}$, hence (a_n) is unbounded. Then we have that

$$a_{b_n - 1} < n \leq a_{b_n}$$

for all $n \in \mathbb{N}$ for which $b_n > 1$. Now, since (a_n) has moderate growth order, it follows that the sequence (a_{n+1}/a_n) has constant growth order. Suppose that it is bounded above by the constant $C > 0$. Then we have that, for all $n \in \mathbb{N}$ for which $b_n > 1$,

$$\frac{a_{b_n}}{a_{b_n-1}} \leq C$$

This means that

$$a_{b_n} \leq C a_{b_n-1} < Cn$$

and therefore we have

$$n \leq a_{b_n} \leq Cn$$

so that $[(a_{b_n})] = \mathfrak{a} \circ \mathfrak{b} = \mathfrak{n}$, and therefore $\mathfrak{b}$ is a right-inverse of $\mathfrak{a}$. □

It is not true, however, that all moderate and monotone growth orders have a *left-inverse*. A simple example is the growth order $\mathfrak{l}$. We can argue that $\mathfrak{l}$ cannot have a left inverse in a few different ways.

Perhaps the simplest argument relies on the observation that $\mathfrak{l} \circ \mathfrak{n} = \mathfrak{l} \circ \mathfrak{n}^2$. If $\mathfrak{l}$ were to have a left-inverse $\mathfrak{a}$, then this would imply $\mathfrak{a} \circ \mathfrak{l} \circ \mathfrak{n} = \mathfrak{a} \circ \mathfrak{l} \circ \mathfrak{n}^2$, and therefore $\mathfrak{n} = \mathfrak{n}^2$, which is a contradiction. In essence, $\mathfrak{l}$ cannot have a left inverse if left-composition $\mathfrak{l} \circ -$ fails to be injective.

We could also make a bounding argument. If $\mathfrak{a}$ were a left-inverse of $\mathfrak{l}$, then it would not be possible for $\mathfrak{a} \leq \mathfrak{n}^p$ for any $p > 0$, because this would imply $\mathfrak{a} \circ \mathfrak{l} \leq \mathfrak{l}^p < \mathfrak{n}$. But in order for the composition $\mathfrak{a} \circ \mathfrak{l}$ to be defined at all, we would need $\mathfrak{a}$ to be moderate, and it is known that all moderate growth orders are $\leq \mathfrak{n}^p$ for some $p > 0$, by Proposition 6. Hence, $\mathfrak{l}$ cannot have a left-inverse.

However, under more stringent conditions than moderateness and monotonicity, we can guarantee the existence of inverses. We will need to prove the following lemma first:

Proposition 71. If $\mathfrak{a}$ is moderate and the quotients $\mathfrak{n}^p/\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{n}^q$ are both monotone increasing for some $p, q > 0$, then $\mathfrak{a}$ has a moderate right-inverse $\mathfrak{b}$ such that $\mathfrak{n}^{1/q}/\mathfrak{b}$ and $\mathfrak{b}/\mathfrak{n}^{1/p}$ are also monotone increasing.

Proof. Following the same strategy as in Proposition 70, let $(a_n) \in \mathfrak{a}$ be chosen such that a_n/n^q (and hence a_n) is monotone increasing, and define (b_n) by letting b_n equal the least integer m such that $a_m \geq n$. We showed in Proposition 70 that $(a_{b_n}) \in \mathfrak{n}$, so that $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{n}$ and $\mathfrak{b}$ is indeed a right-inverse of $\mathfrak{a}$. The rest of the claim remains to be proven.

First we show that $\mathfrak{b}$ is moderate. The sequence (b_n) is monotone increasing by its definition. If $n \leq m \leq 2n$, we have that $b_n \leq b_m$ by monotonicity, so that to prove moderateness it suffices to show that $b_m \leq K b_n$ for all such $m, n \in \mathbb{N}$, or to show that $b_{2n} \leq K b_n$ for all $n \in \mathbb{N}$, for some constant $K > 0$, using Proposition 7. It suffices to show that $a_{Kn} \geq 2a_n$ for all $n \in \mathbb{N}$, for some $K \in \mathbb{N}$. Since (a_n) was chosen to make a_n/n^q monotone increasing, we have that for any $K \in \mathbb{N}$,

$$\frac{a_{Kn}/(Kn)^q}{a_n/n^q} \geq 1$$

or equivalently

$$a_{Kn} \geq K^q a_n$$

Hence, if we choose $K = \lceil 2^{1/q} \rceil$, we will have that $a_{Kn} \geq 2a_n$ as desired, proving that $b_{2n} \leq Kb_n$ and therefore (b_n) and $\mathfrak{b}$ are moderate as claimed.

We have shown that $\mathfrak{b}$ is a right-inverse of $\mathfrak{a}$ and is moderate and monotone. Now, since (a_n/n^q) is monotone increasing and (b_n) is also monotone increasing, it follows that (a_{b_n}/b_n^q) is monotone increasing, and this sequence has growth order $\mathfrak{n}/\mathfrak{b}^q$. Thus, $\mathfrak{n}/\mathfrak{b}^q$ is a monotone increasing growth order, and so is $(\mathfrak{n}/\mathfrak{b}^q)^{1/q} = \mathfrak{n}^{1/q}/\mathfrak{b}$, as claimed.

Finally, since the growth order $\mathfrak{n}^p/\mathfrak{a}$ is monotone by assumption, we may let $(r_n) \in \mathfrak{n}^p/\mathfrak{a}$ be monotone increasing, so that the sequence $(a'_n) = (n^p/r_n)$ has a growth order of $\mathfrak{a}$ (and is hence moderate). Now, we have that the sequence $(n^p/a'_n) = (r_n)$ is monotone increasing, and since (b_n) is also monotone increasing, it follows that (b_n^p/a'_{b_n}) is monotone increasing. But this sequence has growth order $\mathfrak{b}^p/(\mathfrak{a} \circ \mathfrak{b}) = \mathfrak{b}^p/\mathfrak{n}$, meaning that $\mathfrak{b}^p/\mathfrak{n}$ is monotone increasing, and $(\mathfrak{b}^p/\mathfrak{n})^{1/p} = \mathfrak{b}/\mathfrak{n}^{1/p}$ is monotone increasing as claimed. □

Theorem 72. *If $\mathfrak{a}$ is moderate and the quotients $\mathfrak{n}^p/\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{n}^q$ are both monotone increasing for some $p, q > 0$, then there is a growth order $\mathfrak{b}$ which is both the unique left-inverse and the unique right-inverse of $\mathfrak{a}$.*

Proof. Suppose $\mathfrak{a}$ is moderate and the quotients $\mathfrak{n}^p/\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{n}^q$ are monotone increasing for some $p, q > 0$. Then, by Proposition 71, it has a moderate right-inverse $\mathfrak{b}$ such that $\mathfrak{n}^{1/q}/\mathfrak{b}$ and $\mathfrak{b}/\mathfrak{n}^{1/p}$ are monotone increasing. Applying Proposition 71 once more, we have that $\mathfrak{b}$ *also* has a moderate right-inverse $\mathfrak{a}'$. Hence, we have $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{b} \circ \mathfrak{a}' = \mathfrak{n}$, with all three growth orders moderate and $> \mathbf{1}$ so that all of their pairwise composites are defined.

This means that $\mathfrak{b}$ has both a left-inverse $\mathfrak{a}$ and a right-inverse $\mathfrak{a}'$, so that, by Proposition 69, we have $\mathfrak{a} = \mathfrak{a}'$ and therefore $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{b} \circ \mathfrak{a} = \mathfrak{n}$. This inverse is necessarily unique: for if $\mathfrak{a}$ has a growth order $\mathfrak{c}$ as a left- or right-inverse, then Proposition 69 would again imply $\mathfrak{c} = \mathfrak{b}$, since $\mathfrak{b}$ is an inverse on both sides of $\mathfrak{a}$. □

Question 5 Does there exist a moderate growth order $\mathfrak{a}$ such that $\mathfrak{n}^p/\mathfrak{a}$ is not monotone for any $p > 0$?

Because of the existence and uniqueness components of Theorem 72, we are justified in making the following definition:

Definition 73. If $\mathfrak{a}$ is a moderate growth order such that $\mathfrak{n}^p/\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{n}^q$ are monotone increasing for some $p, q > 0$, then we may define $\mathrm{inv}(\mathfrak{a})$ to be the unique left- and right-inverse, or just the **inverse**, of $\mathfrak{a}$ under composition.

Let us now attempt to calculate the inverses of some commonly-encountered growth orders. An easy starting point would be to consider the growth orders $\mathfrak{n}^p$ with $p > 0$, which clearly have as their inverses the growth orders $\mathfrak{n}^{1/p}$. Thus, we may write

$$\mathrm{inv}(\mathfrak{n}^p) = \mathfrak{n}^{1/p}$$

This was an easy calculation, because we know an explicit formula for the inverse of the real-valued function $x \mapsto x^p$, namely $x \mapsto x^{1/p}$.

A trickier example, however, is the growth order $\mathfrak{n}\mathfrak{l}$. How can we compute the inverse of this growth order? The trick lies in noticing that $\mathfrak{l}$ absorbs a lot of growth orders - in particular, we can say that $\mathfrak{l}$ absorbs any growth order that is bounded between $\mathfrak{n}^p$ and $\mathfrak{n}^q$ for any $q > p > 0$, which encompasses all moderate growth orders exhibiting greater than sub-polynomial growth. If $\mathfrak{n}\mathfrak{l}$ has an inverse, it cannot be sub-polynomial, for these growth orders have no left-inverses. Hence, whatever the inverse $\mathfrak{a}$ of $\mathfrak{n}\mathfrak{l}$ must be, it is absorbed by $\mathfrak{l}$. This allows us to conclude that

$$\mathfrak{n}\mathfrak{l} \circ \mathfrak{a} = (\mathfrak{n} \circ \mathfrak{a})\mathfrak{l} = \mathfrak{a}\mathfrak{l}$$

but if $\mathfrak{a}$ is to be an inverse of $\mathfrak{n}\mathfrak{l}$, the above growth order should equal $\mathfrak{n}$. Thus, we have $\mathfrak{a}\mathfrak{l} = \mathfrak{n}$, and therefore $\mathfrak{a} = \mathfrak{n}/\mathfrak{l}$. We have therefore calculated

$$\mathrm{inv}(\mathfrak{n}\mathfrak{l}) = \mathfrak{n}/\mathfrak{l}$$

The same trick can be applied in many cases - for instance, it also works for any product of powers of nested logarithms times $\mathfrak{n}$. For instance,

$$\mathrm{inv}\left(\frac{\mathfrak{n}\mathfrak{l}_3^{10}}{\mathfrak{l}\mathfrak{l}_2^2}\right) = \frac{\mathfrak{n}\mathfrak{l}\mathfrak{l}_2^2}{\mathfrak{l}_3^{10}}$$

What about a *power* of $\mathfrak{n}$ times some product of nested logarithms, for instance $\mathfrak{n}^2\mathfrak{l}$? This can be computed using a similar trick. Again, if $\mathfrak{a}$ is an inverse of this growth order, it must be absorbed by $\mathfrak{l}$, so we would have

$$\mathfrak{n} = \mathfrak{n}^2\mathfrak{l} \circ \mathfrak{a} = \mathfrak{a}^2\mathfrak{l}$$

implying that $\mathfrak{a}^2 = \mathfrak{n}/\mathfrak{l}$ and therefore $\mathfrak{a} = \sqrt{\mathfrak{n}/\mathfrak{l}}$. Hence

$$\mathrm{inv}(\mathfrak{n}^2\mathfrak{l}) = \sqrt{\mathfrak{n}/\mathfrak{l}}$$

It is not difficult to see that this trick also works for growth orders taking the form $\mathfrak{n}\mathfrak{l}^p$ for any exponent p, or more generally, for any moderate growth order taking the form $\mathfrak{n}\mathfrak{a}$ where $\mathfrak{a}$ absorbs any moderate growth order. A sufficient condition for this to be the case is for $\mathfrak{l}^p/\mathfrak{a}$ and

$\mathfrak{a}$ to be monotone increasing for some $p \in \mathbb{R}^+$. But we can actually generalize this trick even further to growth orders that "are not as absorptive".

Theorem 74. *Let $\mathfrak{q}_{1/2}$ be the growth order of $(e^{\sqrt{\log n}})$. If $\mathfrak{a}$ is a moderate growth order such that $\mathfrak{a}$ and $\mathfrak{q}_{1/2}^p/\mathfrak{a}$ are monotone increasing for some $p > 0$, then $\mathrm{inv}(\mathfrak{n}\mathfrak{a}) = \mathfrak{n}/\mathfrak{a}$.*

Proof. We can start by proving that under these hypotheses, $\mathfrak{q}_{1/2}$ absorbs $\mathfrak{n}\mathfrak{a}$. To start with, notice that if $(a_n) \in \mathfrak{a}$, then we have

$$\begin{aligned}
\sqrt{\log(na_n)} &= \sqrt{\log n + \log a_n} \\
&= \sqrt{\log n} \cdot \sqrt{1 + \frac{\log a_n}{\log n}} \\
&= \sqrt{\log n} \cdot \left(1 + O\left(\frac{\log a_n}{\log n}\right)\right) \\
&= \sqrt{\log n} + O\left(\frac{\log a_n}{\sqrt{\log n}}\right) \\
&= \sqrt{\log n} + O\left(\frac{\log(a_n/n)}{\sqrt{\log n}} + 1\right) \\
&= \sqrt{\log n} + O(1)
\end{aligned}$$

This implies that $\sqrt{\log(na_n)} - \sqrt{\log n}$ is bounded, and therefore

$$\left[\exp p\sqrt{\log(na_n)}\right] = \left[\exp p\sqrt{\log n}\right]$$

This in turn implies that $\mathfrak{q}_{1/2}$ absorbs $\mathfrak{n}\mathfrak{a}$. But because $\mathfrak{a}$ and $\mathfrak{q}_{1/2}/\mathfrak{a}$ are both monotone increasing, by Proposition 66, we have that $\mathfrak{a}$ also absorbs $\mathfrak{n}\mathfrak{a}$. Therefore $\mathfrak{a} \circ \mathfrak{n}\mathfrak{a} = \mathfrak{a}$, and hence $(\mathfrak{n}/\mathfrak{a}) \circ \mathfrak{n}\mathfrak{a} = \mathfrak{n}$ and $\mathfrak{n}\mathfrak{a} \circ (\mathfrak{n}/\mathfrak{a}) = \mathfrak{n}$, meaning that $\mathrm{inv}(\mathfrak{n}\mathfrak{a}) = \mathfrak{n}/\mathfrak{a}$ as claimed. □

This is a peculiar "cutoff" between sequences that permit this simple inversion trick and sequences that do not. In fact, for growth orders $\mathfrak{a}$ that are not absorbed by $\mathfrak{q}_{1/2}$, the problem of inverting $\mathfrak{n}\mathfrak{a}$ can get quite nasty. One can verify by direct calculation that the inverse of the growth order $[n \cdot \exp(\log n)^q]$ for values of $q \in (1/2, 2/3]$ is given by

$$\left[\frac{n \cdot \exp q(\log n)^{2q-1}}{\exp(\log n)^q}\right]$$

and for values of q in the interval $(2/3, 1)$ the computation becomes even more cumbersome.

Question 6 The author has conjectured that if $\mathfrak{a}/\mathfrak{b}$ is monotone increasing for some $\mathfrak{a}, \mathfrak{b} > \mathfrak{n}$, and $\mathrm{inv}(\mathfrak{a})$ and $\mathrm{inv}(\mathfrak{b})$ are defined, then $\mathfrak{a} \cdot \mathrm{inv}(\mathfrak{a}) \geq \mathfrak{b} \cdot \mathrm{inv}(\mathfrak{b})$. Is this true?

Proposition 75. For all growth orders $\mathfrak{a}$ for which $\mathfrak{a}/\mathfrak{n}$ is monotone and $\mathrm{inv}(\mathfrak{a})$ is defined, we have that
$$\mathfrak{a} \cdot \mathrm{inv}(\mathfrak{a}) \geq \mathfrak{n}^2$$

Proof. First of all, suppose that $\mathfrak{a}/\mathfrak{n}$ is monotone increasing, so that $\mathfrak{n} \leq \mathfrak{a}$ and $\mathrm{inv}(\mathfrak{a}) \leq \mathfrak{n}$. Then by Proposition 62 we have that
$$\frac{\mathfrak{a}}{\mathfrak{n}} \circ \mathrm{inv}(\mathfrak{a}) \leq \frac{\mathfrak{a}}{\mathfrak{n}}$$
or
$$\frac{\mathfrak{a} \circ \mathrm{inv}(\mathfrak{a})}{\mathfrak{n} \circ \mathrm{inv}(\mathfrak{a})} = \frac{\mathfrak{n}}{\mathrm{inv}(\mathfrak{a})} \leq \frac{\mathfrak{a}}{\mathfrak{n}}$$
By multiplying both sides of this inequality by $\mathfrak{n} \cdot \mathrm{inv}(\mathfrak{a})$ we obtain the desired inequality
$$\mathfrak{n}^2 \leq \mathfrak{a} \cdot \mathrm{inv}(\mathfrak{a})$$
For the case in which $\mathfrak{a}/\mathfrak{n}$ is monotone decreasing, we have $\mathfrak{n} \geq \mathfrak{a}$ and $\mathrm{inv}(\mathfrak{a}) \geq \mathfrak{n}$, so that
$$\frac{\mathfrak{a}}{\mathfrak{n}} \circ \mathrm{inv}(\mathfrak{a}) \leq \frac{\mathfrak{a}}{\mathfrak{n}}$$
yet again, and the same algebraic manipulation as before leads to the desired result. $\square$

Notice that when $\mathfrak{a} = \mathfrak{n}^p$ is a power function, we have
$$\mathfrak{a} \cdot \mathrm{inv}(\mathfrak{a}) = \mathfrak{n}^{p+\frac{1}{p}}$$
so that in this case, the proposition reduces to the well-known inequality
$$p + \frac{1}{p} \geq 2$$
for real numbers $p > 0$.

Now, let us briefly look at a few applications of inversion. An example of a possible application area would be in analytic number theory, in which the asymptotic growth orders of certain sequences of positive integers might be of interest. In particular, if (a_n) is a monotone increasing sequence of positive integers with growth order $\mathfrak{a}$, then $\mathrm{inv}(\mathfrak{a})$ tells us the growth order of the "counting sequence" $|\{a_k : a_k \leq n, k \in \mathbb{N}\}|$ which tracks the number of elements of (a_n) under some given value. For instance, if we know that $\mathrm{inv}(\mathfrak{n}/\mathfrak{l}) = \mathfrak{n}\mathfrak{l}$, and we know that the number of primes $\pi(n)$ beneath a positive integer n has growth order $\Theta(n/\log n)$, then we can say immediately that p_n, the nth prime number, has growth order $\Theta(n \log n)$, or vice versa.

Let us consider a more sophisticated example: suppose we are interested in the sequence of positive integers (a_n) which can be written as a sum of two cubes, and wish to know the asymptotic growth order of this sequence of integers. Let $c(n)$ be the "counting function" for this sequence, so that $c(n)$ is the number of elements of (a_n) below n. There are also only $\Theta(n^{1/3})$ perfect cubes under n, meaning that there are only $\Theta(n^{2/3})$ *pairs* of cubes under n, so we can argue that $c(n) = O(n^{2/3})$. Thus, (a_n) has growth order upper-bounded by the inverse of $\mathfrak{n}^{2/3}$, meaning that its growth order is lower-bounded by $\mathfrak{n}^{3/2}$, or $a_n = \Omega(n^{3/2})$. By similar reasoning, if we were instead interested in the sequence of positive integers which are the sum of a square and a cube, we would come up with the lower bound $\Omega(n^{6/5})$.

Here is another interesting bounding argument that makes use of the pigeonhole principle. Suppose that $f(n)$ counts the number of ways to write n as a sum of 3 prime numbers. Beneath $n/3$, there are $\Theta(n/\log n)$ prime numbers, meaning that there are at least $\Omega(n^3/\log^3 n)$ distinct triples of prime numbers whose sums are at most n. Hence, by pigeonhole, there must be an infinite sequence of natural numbers n such that $f(n)$ is $\Omega(n^2/\log^3 n)$. That is,

$$\sup_{1 \le k \le n} f(k) = \Omega\left(\frac{n^2}{\log^3 n}\right)$$

If a sequence (b_n) is defined such that b_n is the smallest positive integer that can be written as a sum of 3 primes in n distinct ways, then we have that

$$b_n = O\left(\sqrt{n \log^3 n}\right)$$

since $\mathrm{inv}(\mathfrak{n}^2/\mathfrak{l}^3) = \sqrt{\mathfrak{n}\mathfrak{l}^3}$. Examples like these show the utility of being able to convert with ease between a growth order and its compositional inverse.

5.5. Composition groups

If we find a set of growth orders that is both closed under composition *and* contains an inverse for each of its elements, then it carries the structure of not only a monoid, but a *group*. For instance, consider the set of power growth orders with a positive power, taking the form $\mathfrak{n}^p$ with $p > 0$. We have that $\mathfrak{n}^p \circ \mathfrak{n}^q = \mathfrak{n}^{pq}$ and $\mathrm{inv}(\mathfrak{n}^p) = \mathfrak{n}^{1/p}$, so that this set of growth orders has the same group structure as $\mathbb{R}_{\times}^{>0}$, the group of positive real numbers under multiplication, which is isomorphic to the group $\mathbb{R}_{+}$ of real numbers under addition.

We can also consider the group of growth orders taking the form $\mathfrak{n}^p\mathfrak{l}^q$, where $p > 0$ and q is any real number. If we represent elements of this group by tuples (p, q), then the group law of this set of growth orders is given by

$$(p, q) \circ (r, s) = (pr, ps + q)$$

and the formula for the inverse of an element is

$$(p, q)^{-1} = (p^{-1}, -q/p)$$

Interestingly, this group is actually isomorphic to a group of 2×2 upper-triangular matrices. The embedding is given as follows:

$$\varphi(p, q) = \begin{bmatrix} p & q \\ 0 & 1 \end{bmatrix}$$

which might make one wonder whether other interesting composition groups of growth orders have matrix representations. Sadly, this is a question the author did not have time to fully explore as part of this thesis.

We can form one large group of growth orders which contains all of the composition groups that we will consider as its subgroups. This group will consist of all growth orders meeting the conditions for the existence of composites and an inverse:

Definition 76. Let $C \subset \mathcal{S}(\mathbb{R}^+)/\sim$ denote the set of all moderate growth orders $\mathfrak{c}$ such that $\mathfrak{n}^p/\mathfrak{c}$ and $\mathfrak{a}/\mathfrak{n}^q$ are both monotone increasing for some $p, q > 0$. This set will be called the **maximal composition group**.

Proposition 77. C is a group under composition of growth orders.

Proof. Firstly, we must verify that C is closed under composition. Suppose $\mathfrak{a}, \mathfrak{b} \in C$ so that both growth orders are moderate and $\mathfrak{n}^p/\mathfrak{a}, \mathfrak{n}^{p'}/\mathfrak{b}, \mathfrak{a}/\mathfrak{n}^q, \mathfrak{b}/\mathfrak{n}^{q'}$ are monotone increasing. We have shown in Proposition 58 that $\mathfrak{a} \circ \mathfrak{b}$ must also be moderate. Additionally, since $\mathfrak{n}^p/\mathfrak{a}$ is monotone increasing and $\mathfrak{b}/\mathfrak{n}^{q'}$ is monotone increasing, we have that $\mathfrak{b}$ is monotone increasing, and

$$\frac{\mathfrak{n}^p}{\mathfrak{a}} \circ \mathfrak{b} = \frac{\mathfrak{b}^q}{\mathfrak{a} \circ \mathfrak{b}}$$

is monotone increasing. Additionally, since $\mathfrak{n}^{p'}/\mathfrak{b}$ is monotone increasing, we have that $\mathfrak{n}^{p'q}/\mathfrak{b}^q$ is monotone increasing, and therefore

$$\frac{\mathfrak{b}^q}{\mathfrak{a} \circ \mathfrak{b}} \cdot \frac{\mathfrak{n}^{p'q}}{\mathfrak{b}^q} = \frac{\mathfrak{n}^{p'q}}{\mathfrak{a} \circ \mathfrak{b}}$$

is monotone increasing. Hence, letting $r = p'q$ ensures that $\mathfrak{n}^r/\mathfrak{a} \circ \mathfrak{b}$ is monotone increasing. By similar reasoning, we may show that letting $s = pq'$ ensures that $\mathfrak{a} \circ \mathfrak{b}/\mathfrak{n}^s$ is monotone increasing. Thus, C is closed under composition.

To see why every element of C has an inverse in C, we may use Proposition 71 and Theorem 72. □

Theorem 78. *C is a lattice-ordered group. (That is, it is an ordered group in which least upper bounds and greatest lower bounds exist, such that the group operation is order-preserving in both arguments.)*

Proof. We have shown above that C is a group, so it remains to show that it is a lattice, and that the group operation $\circ$ interacts with the partial ordering as required.

We demonstrated in Proposition 29 that $\mathcal{S}(\mathbb{R}^+)/\sim$ is a lattice with join $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{a} + \mathfrak{b}$ and meet $\mathfrak{a} \wedge \mathfrak{b} = (\mathfrak{a}^{-1} + \mathfrak{b}^{-1})^{-1}$. Thus, since $C \subset \mathcal{S}(\mathbb{R}^+)/\sim$, it suffices to show that $\mathfrak{a} \vee \mathfrak{b} \in C$ and

$\mathfrak{a} \wedge \mathfrak{b} \in C$ for all $\mathfrak{a}, \mathfrak{b} \in C$, since this implies that $\vee$ and $\wedge$ are also join and meet operations on C, making it a lattice. Hence, let $\mathfrak{a}, \mathfrak{b} \in C$ be arbitrary, so that $\mathfrak{a}, \mathfrak{b}$ are moderate and $\mathfrak{n}^p/\mathfrak{a}, \mathfrak{n}^{p'}/\mathfrak{b}, \mathfrak{a}/\mathfrak{n}^q, \mathfrak{b}/\mathfrak{n}^{q'}$ are monotone increasing for some $p, p', q, q' > 0$. We know that taking sums and inverses preserves moderate growth, so we certainly have that $\mathfrak{a} \vee \mathfrak{b}$ and $\mathfrak{a} \wedge \mathfrak{b}$ are moderate. Additionally, we have that $\mathfrak{a}/\mathfrak{n}^{\min(q,q')}$ and $\mathfrak{b}/\mathfrak{n}^{\min(q,q')}$ are both monotone increasing, thus $(\mathfrak{a} + \mathfrak{b})/\mathfrak{n}^{\min(q,q')} = (\mathfrak{a} \vee \mathfrak{b})/\mathfrak{n}^{\min(q,q')}$ is monotone increasing. Additionally, $\mathfrak{a}/\mathfrak{n}^{\max(p,p')}$ and $\mathfrak{b}/\mathfrak{n}^{\max(p,p')}$ are both monotone decreasing, so that $(\mathfrak{a} + \mathfrak{b})/\mathfrak{n}^{\max(p,p')}$, or $(\mathfrak{a} \vee \mathfrak{b})/\mathfrak{n}^{\max(p,p')}$, is also monotone decreasing, and $\mathfrak{n}^{\max(p,p')}/(\mathfrak{a} \vee \mathfrak{b})$ is monotone increasing. Hence, $\mathfrak{a} \vee \mathfrak{b} \in C$. Similar reasoning also shows that $\mathfrak{a} \wedge \mathfrak{b} \in C$, in particular by showing that $\mathfrak{n}^{\max(p,p')}/(\mathfrak{a} \wedge \mathfrak{b})$ and $(\mathfrak{a} \wedge \mathfrak{b})/\mathfrak{n}^{\min(q,q')}$ are also monotone increasing.

Finally, we just need to show that left- and right-composition preserve the order structure of C. This follows directly from Proposition 62, since all growth orders in C are monotone and $> \mathbf{1}$. □

The structure of a lattice-ordered group tells us a lot about the algebraic properties of this collection of growth orders. [2] gives a detailed treatment of the theory of lattice-ordered groups, from which we can deduce several results "for free" about the behavior of the growth orders in C. For instance:

- C is torsion-free. That is, no growth order $\mathfrak{c} \neq \mathfrak{n}$ in C satisfies $\mathfrak{c} \circ \mathfrak{c} \circ \cdots \circ \mathfrak{c} = \mathfrak{n}$. (Proposition 3.5 in Darnel.)
- For any $\mathfrak{a}, \mathfrak{b} \in C$, we have that $\operatorname{inv}(\mathfrak{a} + \mathfrak{b}) = ((\operatorname{inv}\mathfrak{a})^{-1} + (\operatorname{inv}\mathfrak{b})^{-1})^{-1}$, including for incomparable growth orders $\mathfrak{a} \perp \mathfrak{b}$. (Proposition 3.2b in Darnel.)
- If $\mathfrak{a} + \mathfrak{b} = \mathfrak{n}$, then $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{b} \circ \mathfrak{a}$. (Dual of Proposition 3.10 in Darnel.)
- The growth orders $\mathfrak{a} \geq \mathfrak{n}$ generate all of C. (Corollary 2.7 in Darnel.)

Question 7 Every composition group of growth orders generated by one element is isomorphic to $\mathbb{Z}$, the additive group of the integers. (This follows from the torsion-free property mentioned above.) What possible forms can a composition group of growth orders generated by *two* or *three* elements take?

6. Closed chains and exponential extensions

We have already examined a few different subsets of $\mathcal{S}(\mathbb{R}^+)/\sim$, consisting of growth orders subject to certain "niceness" conditions, such as *moderateness* and *monotonicity*. In order to "zoom in" on a particular subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ and do a deeper analysis there, we would like it to satisfy a few criteria. For one, $\mathcal{G}$ should be closed under most of the operations that we would like to perform, such as $+, \cdot, \div$, and Σ. It is often troublesome to deal with $\mathcal{S}(\mathbb{R}^+)/\sim$ because it contains many growth orders with oscillatory behavior that hardly even match our intuition of what a "growth order" should mean, so we would also like $\mathcal{G}$ to exclude many of these pathological sequences. We might even hope for *trichotomy* to hold in $\mathcal{G}$ - that is, for any two growth orders in $\mathcal{G}$ to be comparable, or for $\mathcal{G}$ to be a *chain*.

We have already seen a couple "niceness conditions" that allow us to rule out certain classes of pathological growth orders, namely moderateness and monotonicity. These criteria guarantee some nice things about growth orders, for instance:

- The moderate growth orders are closed under sums, products, quotients, and partial sums (Proposition 19, Proposition 36)
- The monotone growth orders are closed under sums, reciprocals and partial sums
- Every monotone growth order is comparable to $\mathbf{1}$, and consequently any two growth orders with a monotone quotient are comparable to each other (Proposition 40)

However, both of these criteria have caveats as well:

- Moderateness does not guarantee that a growth order is comparable to $\mathbf{1}$ (4.2)
- The monotone growth orders are not pairwise comparable amongst themselves (4.2)
- The monotone growth orders are not closed under products or quotients

Therefore, if we wish to construct a chain $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ which is closed under the operations we care about, we will have to do a bit of fine-tuning of these conditions.

This chapter will bring together all of the operations and properties of growth orders that have appeared in previous chapters, and use them to guarantee that a certain kind of construction always produces chains of growth orders. In Section 6.1 we will devise a collection of properties that guarantees total ordering on a collection of growth orders that is closed under partial summation. In Section 6.2 we will enumerate the growth orders belonging to the simplest possible collection that can result from the former construction. This will involve proving a formula for the partial sums of a very broad class of commonly-encountered growth orders.

Finally, in Section 6.3 we will see how to add more exotic growth orders to this structure without disturbing its favorable properties.

6.1. SR-regularity and closure

Definition 79. We say that a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ is **moderate** if every element of $\mathcal{G}$ is moderate.

Definition 80. We say that a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ has the **monotone quotient property** if every quotient of elements of $\mathcal{G}$ is monotone.

Recall that monotone sequences are always comparable to $\mathbf{1}$, and two sequences are comparable if and only if their quotient is comparable to $\mathbf{1}$. This means that the monotone quotient property guarantees trichotomy in $\mathcal{G}$, and is in fact a much stronger property (as we shall soon see).

Definition 81. We say that a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ is **SR-regular** if, for every element $\mathfrak{a} \in \mathcal{G}$, either $\mathfrak{a}$ or $\mathfrak{a}^{-1}$ has a preimage under Σ in $\mathcal{G}$. That is, for any $\mathfrak{a} \in \mathcal{G}$, there exists $\mathfrak{b} \in \mathcal{G}$ such that either $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$.

Definition 82. Given a subset $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$, denote by $\Sigma\mathcal{G}$ and $\mathrm{R}\mathcal{G}$ the following sets:

$$\Sigma\mathcal{G} = \{\Sigma\mathfrak{a} \ : \ \mathfrak{a} \in \mathcal{G}\}$$

$$\mathrm{R}\mathcal{G} = \{\mathfrak{a}^{-1} \ : \ \mathfrak{a} \in \mathcal{G}\}$$

Proposition 83. If $\mathcal{G}$ is SR-regular, then we have that $\mathcal{G} \subset \Sigma\mathcal{G} \cup \mathrm{R}\Sigma\mathcal{G}$. Further, $\Sigma\mathcal{G} \cup \mathrm{R}\Sigma\mathcal{G}$ is itself SR-regular.

Proof. The former claim follows directly from the definition of SR-regularity. The latter claim follows from the fact that each element $\mathfrak{a} \in \Sigma\mathcal{G} \cup \mathrm{R}\Sigma\mathcal{G}$ takes the form $\Sigma\mathfrak{b}$ or $(\Sigma\mathfrak{b})^{-1}$ for some $\mathfrak{b}$ (by the definition of these sets), hence either $\mathfrak{a}$ or $\mathfrak{a}^{-1}$ has a preimage under Σ contained in $\mathcal{G}$, which is itself contained in $\Sigma\mathcal{G} \cup \mathrm{R}\Sigma\mathcal{G}$ by the former claim. $\square$

The following definition describes how we will expand a set of growth orders $\mathcal{G}$ with many "nice properties" to obtain a broader set of growth orders $\overline{\mathcal{G}}$ with the same "nice properties". In algebraic terms, it is simply the closure of the set $\mathcal{G} \subset \mathcal{S}(\mathbb{R}^+)/\sim$ under the operations Σ and $\mathrm{R}\Sigma$: since we would like a collection of growth orders that is closed under sums and reciprocals, we simply start with $\mathcal{G}$ and repeatedly "add in" the sums and reciprocals of its elements until we arrive at something that is closed under these operations. The hard part of this construction is showing that adding these new growth orders does not break any of the favorable properties of the original set $\mathcal{G}$, which we will prove in the subsequent proposition.

Definition 84. If $\mathcal{G} \in \mathcal{S}(\mathbb{R}^+)/\sim$ is SR-regular, then by Proposition 83 we may recursively define a sequence of nested subsets

$$\mathcal{G} = \mathcal{G}_0 \subset \mathcal{G}_1 \subset \cdots \subset \mathcal{G}_n \subset \cdots$$

by letting $\mathcal{G}_{n+1} = \Sigma\mathcal{G}_n \cup \mathrm{R}\Sigma\mathcal{G}_n$. Define the **SR-closure** of $\mathcal{G}$ to be the set

$$\overline{\mathcal{G}} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$$

The "niceness properties" that we will demand of the starting set $\mathcal{G}$ are as follows:

1. moderateness
2. monotone quotients
3. SR-regularity

Now we will show that when these properties are satisfied in $\mathcal{G}$, they carry over to the expanded set $\overline{\mathcal{G}}$.

Proposition 85. Suppose $\mathcal{G}$ is moderate and SR-regular and satisfies the monotone-quotient property. Then $\overline{\mathcal{G}}$ is also moderate and SR-regular and satisfies the monotone-quotient property, and is furthermore closed under partial sums and reciprocals.

Proof. The fact that $\overline{\mathcal{G}}$ is moderate follows directly from the fact that $\Sigma\mathfrak{a}$ and $\mathfrak{a}^{-1}$ are moderate whenever $\mathfrak{a}$ is moderate.

To prove that $\overline{\mathcal{G}}$ has the monotone-quotient property, we will prove inductively that each $\mathcal{G}_n$ has this property, and since the $\mathcal{G}_n$ are nested and $\overline{\mathcal{G}}$ is their union, it must also have this property. Suppose that all quotients of elements of $\mathcal{G}_n$ are monotone. By the definition of $\mathcal{G}_{n+1}$, each element $\mathfrak{a} \in \mathcal{G}_{n+1}$ is either equal to $\Sigma\mathfrak{a}'$ or $(\Sigma\mathfrak{a}')^{-1}$ for some $\mathfrak{a}' \in \mathcal{G}_n$. Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}_{n+1}$ be arbitrary. WLOG, there are three cases to consider:

(1) $\mathfrak{a} = \Sigma\mathfrak{a}'$ and $\mathfrak{b} = \Sigma\mathfrak{b}'$, with $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$

(2) $\mathfrak{a} = \Sigma\mathfrak{a}'$ and $\mathfrak{b} = (\Sigma\mathfrak{b}')^{-1}$, with $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$

(3) $\mathfrak{a} = (\Sigma\mathfrak{a}')^{-1}$ and $\mathfrak{b} = (\Sigma\mathfrak{b}')^{-1}$, with $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$

In case (2), $\mathfrak{a}$ contains a monotone increasing sequence (a_n) and $\mathfrak{b}$ contains a monotone decreasing sequence (b_n), so that the quotient $\mathfrak{a}/\mathfrak{b}$ contains the monotone increasing sequence (a_n/b_n). In case (1), we have that $\mathfrak{a}/\mathfrak{b} = \Sigma\mathfrak{a}'/\Sigma\mathfrak{b}'$ is monotone by Proposition 43, since $\mathfrak{a}', \mathfrak{b}' \in \mathcal{G}_n$, which has the monotone quotient property by inductive hypothesis. Similarly, in case (3), we have that $\mathfrak{a}/\mathfrak{b} = \Sigma\mathfrak{b}'/\Sigma\mathfrak{a}'$ is monotone. Thus $\mathcal{G}_{n+1}$ has the monotone quotient property, given that $\mathcal{G}_n$ has this property. Since $\mathcal{G}_0$ has this property by hypothesis, we have that each $\mathcal{G}_n$ and therefore $\overline{\mathcal{G}}$ has the monotone quotient property.

To see why $\overline{\mathcal{G}}$ is SR-regular, let $\mathfrak{a} \in \overline{\mathcal{G}}$ be arbitrary so that $\mathfrak{a} \in \mathcal{G}_n$ for some $n \in \mathbb{N}$ or $n = 0$. If $n = 0$, we have that $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$ for some $\mathfrak{b} \in \mathcal{G}_0$ because $\mathcal{G}_0$ is SR-regular by hypothesis. If $n > 0$, then $\mathcal{G}_n = \Sigma\mathcal{G}_{n-1} \cup \mathrm{R}\Sigma\mathcal{G}_{n-1}$, meaning that either $\mathfrak{a} \in \Sigma\mathcal{G}_{n-1}$ or $\mathfrak{a} \in \mathrm{R}\Sigma\mathcal{G}_{n-1}$. In the former case we have $\mathfrak{a} = \Sigma\mathfrak{b}$, and in the latter case we have $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$, where $\mathfrak{b} \in \mathcal{G}_{n-1} \subset \overline{\mathcal{G}}$. In any case, there exists $\mathfrak{b} \in \overline{\mathcal{G}}$ such that $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$, making $\overline{\mathcal{G}}$ SR-regular as claimed.

Finally, we show that $\overline{\mathcal{G}}$ is closed under sums and reciprocals. If $\mathfrak{a} \in \mathcal{G}_n \subset \overline{\mathcal{G}}$, then we have that $\Sigma\mathfrak{a} \in \mathcal{G}_{n+1} \subset \overline{\mathcal{G}}$. Further, we have that either $\mathfrak{a} = \Sigma\mathfrak{b}$ or $\mathfrak{a} = (\Sigma\mathfrak{b})^{-1}$ for some $\mathfrak{b} \in \mathcal{G}_m$. In the former case, we have $\mathfrak{a}^{-1} = (\Sigma\mathfrak{b})^{-1} \in \mathcal{G}_{m+1}$. In the latter case, we have $\mathfrak{a}^{-1} = \Sigma\mathfrak{b} \in \mathcal{G}_{m+1}$. Thus, $\mathfrak{a}^{-1} \in \mathcal{G}_{m+1} \subset \overline{\mathcal{G}}$ in either case, and $\overline{\mathcal{G}}$ is closed under reciprocals. □

Consider the following special set of growth orders:

Definition 86. Let the set of growth orders

$$\mathcal{M} = \{\mathfrak{n}^p : p \in \mathbb{Z}\}$$

be called the **minimal seed set**.

The growth orders $\mathfrak{n}^p$ are both monotone and moderate for all $p \in \mathbb{Z}$. Further, this set is SR-regular because

1. $\mathfrak{n}^p = \Sigma\mathfrak{n}^{p-1}$ for $p = 1, 2, \cdots$
2. $\mathfrak{n}^p = \mathrm{R}\Sigma\mathfrak{n}^{-p-1}$ for $p = -1, -2, \cdots$
3. $\mathfrak{n}^0 = \mathbf{1} = \Sigma\mathfrak{n}^{-2}$ for the case of $p = 0$

Thus, we may consider the SR-closure $\overline{\mathcal{M}}$. This set of growth orders is noteworthy because it is the *smallest* of all SR-closed sets of growth orders!

Proposition 87. If $\mathcal{G}$ is an SR-regular set of growth orders, then $\overline{\mathcal{M}} \subset \overline{\mathcal{G}}$.

Proof. Let $\mathfrak{a} \in \overline{\mathcal{G}}$ be arbitrary. Then we have that $\Sigma\mathfrak{a} \geq \mathbf{1}$, meaning that $\Sigma^3\mathfrak{a} \geq \mathfrak{n}^2$. [1] Thus we have that $\mathrm{R}\Sigma^3\mathfrak{a} \leq \mathfrak{n}^{-2}$ and therefore $\Sigma\mathrm{R}\Sigma^3\mathfrak{a} = \mathbf{1}$. Hence, since $\overline{\mathcal{G}}$ is closed under Σ and R, we have that $\mathbf{1} \in \overline{\mathcal{G}}$.

Finally, since $\mathfrak{n}^p = \Sigma^p\mathbf{1}$ and $\mathfrak{n}^{-p} = \mathrm{R}\Sigma^p\mathbf{1}$ for all $p \in \mathbb{N}$, we have that $\overline{\mathcal{G}}$ contains $\mathfrak{n}^p$ for all $p \in \mathbb{Z}$. Thus, it contains $\mathcal{M} = \mathcal{M}_0$ as a subset, and since it is closed under Σ and R, it follows

[1] Here, we use Σ^3 to denote $\Sigma\Sigma\Sigma$, or $\Sigma \circ \Sigma \circ \Sigma$. Similarly, we will denote by Σ^k the k-fold composition of Σ with itself.

inductively that $\overline{\mathcal{G}}$ contains $\mathcal{M}_n$ for all $n \in \mathbb{N}$ and therefore their union $\overline{\mathcal{M}}$ is also contained in $\overline{\mathcal{G}}$, as claimed. $\square$

6.2. Nested log sums

In this section, we will "get down in the weeds" by taking a closer look at $\overline{\mathcal{M}}$ and completely characterizing all of the growth orders contained in it. This will therefore determine which growth orders are *common to all* SR-closed chains of $\mathcal{S}(\mathbb{R}^+)/\sim$. The answer is rather surprising: the growth orders contained in $\overline{\mathcal{M}}$ are precisely those taking the form

$$\mathfrak{l}(p_0, \cdots, p_m) = \mathfrak{n}^{p_0}\mathfrak{l}_1^{p_1} \cdots \mathfrak{l}_m^{p_m}$$

for $p_0, \cdots, p_m \in \mathbb{Z}$. This has a shocking implication: any collection of sequences closed under both partial sums and reciprocals contains sequences with the growth order of

$$n^{p_0} \cdot (\log n)^{p_1} \cdot \cdots \cdot (\overbrace{\log \cdots \log}^{m \text{ nested logs}} n)^{p_m}$$

for any choice of $m \in \mathbb{N}$ and $p_0, \cdots, p_m \in \mathbb{Z}$. The moral of this story is: if you want to study growth orders in an environment that allows you to take partial sums and reciprocals of sequences, you cannot avoid dealing with strange growth orders involving nested logarithms!

Before tackling this, we must prove several "niceness" properties for growth orders taking the above form. Along the way, we will learn an explicit formula for the partial sums of any product of powers of nested logarithms.

Proposition 88. For all $m \in \mathbb{N}$, the growth order $\mathfrak{l}_m$ is moderate and monotone.

Proof. We know $\mathfrak{l} = \mathfrak{l}_1$ is moderate and monotone, since it equals $\Sigma\mathfrak{n}^{-1}$. Since the composition of two moderate and monotone growth orders is also moderate and monotone, and $\mathfrak{l}_{m+1} = \mathfrak{l}_m \circ \mathfrak{l}$, the desired result follows by induction. $\square$

Proposition 89. For all $p \in \mathbb{R}^+$ and $i, j \in \mathbb{N}$ with $i < j$, we have $\mathfrak{l}_j^p \leq \mathfrak{l}_i$. Additionally, $\mathfrak{l}_i^p \leq \mathfrak{n}$ for all $p \in \mathbb{R}^+$ and $i \in \mathbb{N}$.

Proof. Let $(a_n) \in \mathfrak{l}_i$ be such that $a_n > 1$ for all $n \in \mathbb{N}$, so that the sequence $(\log a_n)$ has growth order $\mathfrak{l}_{i+1}$. Since $\log x \leq x$ for all $x \in \mathbb{R}^+$, we have that $\log a_n^{1/p} \leq a_n^{1/p}$ for all $n \in \mathbb{N}$ and $p \in \mathbb{R}^+$, or equivalently $(\log a_n)^p \leq p^p a_n$. The LHS of this inequality has growth order $\mathfrak{l}_{i+1}^p$ and the RHS has growth order $\mathfrak{l}_i$, so we have that $\mathfrak{l}_{i+1}^p \leq \mathfrak{l}_i$ for all $p \in \mathbb{R}^+$ and $i \in \mathbb{N}$. This means that if $i < j$, we have

$$\mathfrak{l}_j^p \leq \mathfrak{l}_{j-1} \leq \cdots \leq \mathfrak{l}_{i+1} \leq \mathfrak{l}_i \leq \mathfrak{n}$$

and therefore $\mathfrak{l}_j^p \leq \mathfrak{l}_i \leq \mathfrak{n}$, as desired. $\square$

We are now ready to show that all of the growth orders taking the form $\mathfrak{l}(p_0, \cdots, p_m)$ are monotone.

Proposition 90. For all $p_0, \cdots, p_m \in \mathbb{R}$, the growth order $\mathfrak{l}(p_0, \cdots, p_m)$ is monotone.

Proof. We proceed by induction on m. Suppose that this is true for some $m \in \mathbb{N}$. Since $\mathfrak{l}(q_0, \cdots, q_m)$ is monotone for all $q_0, \cdots, q_m \in \mathbb{R}$, and $\mathfrak{l} > 1$ is also monotone, we have that the growth order

$$\mathfrak{l}(q_0, \cdots, q_m) \circ \mathfrak{l} = \mathfrak{l}(0, q_0, \cdots, q_m)$$

is monotone by Proposition 59. Now notice that, since $\mathfrak{l}$ absorbs $\mathfrak{n}^{1/2}$, we also have that $\mathfrak{l}(0, q_0, \cdots, q_m)$ absorbs $\mathfrak{n}^{1/2}$. Further, by Proposition 89, we have that $\mathfrak{l}(0, q_0, \cdots, q_m)^{-2} \leq \mathfrak{n}$, which, by raising both sides to the $1/2$ power and multiplying across by $\mathfrak{n}^{1/2}\mathfrak{l}(0, q_0, \cdots, q_m)$, implies $\mathfrak{n}^{1/2} \leq \mathfrak{n}\mathfrak{l}(0, q_0, \cdots, q_m)$. Therefore, by Proposition 67, we have

$$\Sigma\mathfrak{l}(0, q_0, \cdots, q_m) = \mathfrak{n}\mathfrak{l}(0, q_0, \cdots, q_m) = \mathfrak{l}(1, q_0, \cdots, q_m)$$

which is monotone because it is a partial sum. This further implies that $\mathfrak{l}(1, q_0, \cdots, q_m)^r$, or $\mathfrak{l}(r, rq_0, \cdots, rq_m)$, is monotone, for any $r \in \mathbb{R}$. It follows that $\mathfrak{l}(p_0, \cdots, p_{m+1})$ is monotone for any $p_0, \cdots, p_{m+1} \in \mathbb{R}$, since the case of $p_0 = 0$ has already been considered, and any tuple $(p_0, \cdots, p_{m+1})$ with $p_0 \neq 0$ can be written in the form $(r, rq_0, \cdots, rq_m)$ for some $r, q_0, \cdots, q_m \in \mathbb{R}$. (In particular, we would set $r = p_0$ and $q_i = p_{i+1}/p_0$ for each i.) Thus, the inductive step is proven.

The base case of $m = 0$ is clearly true, since $\mathfrak{l}(p_0) = \mathfrak{n}^{p_0}$ is monotone for any $p_0 \in \mathbb{R}$. Hence, by induction, we have that $\mathfrak{l}(p_0, \cdots, p_m)$ is monotone for any $p_0, \cdots, p_m \in \mathbb{R}$. □

Having proven all of the "niceness" properties we need for nested logarithms, we are now ready to start figuring out how to calculate their partial sums. The next proposition proves the following family of asymptotic identities:

$$\sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$$

$$\sum_{k=2}^{n} \frac{1}{k \log k} = \Theta(\log \log n)$$

$$\sum_{k=3}^{n} \frac{1}{k \log k \log \log k} = \Theta(\log \log \log n)$$

Proposition 91. For all $m \in \mathbb{N}$, we have

$$\Sigma(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1} = \mathfrak{l}_{m+1}$$

Proof. We will prove this by induction. Suppose that this claim is true for some $m \in \mathbb{N}$. Then we have

$$\mathrm{P}(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1} = \frac{(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}}{\mathfrak{l}_{m+1}} = (\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m \mathfrak{l}_{m+1})^{-1}$$

From Proposition 90, we also have that $(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$ is moderate. Therefore, by Proposition 48, the partial sum of $\mathrm{P}(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$ has the growth order of $\mathfrak{l} \circ \mathfrak{l}_{m+1}$, or $\mathfrak{l}_{m+2}$. This means that

$$\Sigma \mathrm{P}(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1} = \Sigma(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m \mathfrak{l}_{m+1})^{-1} = \mathfrak{l}_{m+2}$$

and therefore the inductive step is completed. The base case of $m = 0$ is true because $\Sigma \mathfrak{n}^{-1} = \mathfrak{l}_1$, and thus the claim is proven. □

Finally, we are ready to generalize to all products of powers of nested logarithms:

Theorem 92. *Let $p_l, \cdots, p_m \in \mathbb{R}$ with $p_l \neq -1$. Then we have that*

$$\Sigma \mathfrak{l}(-1, \cdots, -1, p_l, p_{l+1}, \cdots, p_m) = \begin{cases} \mathfrak{l}(0, \cdots, 0, p_l + 1, p_{l+1}, \cdots, p_m) & \text{if } p_l > -1 \\ \mathbf{1} & \text{if } p_l < -1 \end{cases}$$

Proof. First of all, suppose that $p_l < -1$. (We shall handle the second case first.) By Proposition 89, we have that

$$\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m) < \mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)$$

for any $\epsilon > 0$. By choosing $\epsilon < -(p_l + 1)$, we can ensure that $p_l + \epsilon < -1$. Now, Proposition 91 implies that

$$\mathfrak{l}(-1, \cdots, -1, p_l + \epsilon) = \mathfrak{l}(-1, \cdots, -1) \cdot (\Sigma \mathfrak{l}(-1, \cdots, -1))^{p_l + \epsilon}$$

and from here, Proposition 49 implies that $\Sigma \mathfrak{l}(-1, \cdots, -1, p_l + \epsilon) = \mathbf{1}$, since $p_l + \epsilon < -1$. Since $\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m)$ is less than $\mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)$, its partial sum also converges, and therefore

$$\Sigma \mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m) = \mathbf{1}$$

which proves the second case.

Now suppose that $p_l > -1$. By Proposition 89, we have that

$$\mathfrak{l}(-1, \cdots, -1, p_l - \epsilon) \leq \mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m) \leq \mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)$$

for any $\epsilon > 0$. Let us choose $\epsilon < p_l + 1$ so that $p_l - \epsilon > -1$. Now, the ratios

$$\frac{\mathfrak{l}(-1, \cdots, -1, p_l + \epsilon)}{\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m)}$$

and

$$\frac{\mathfrak{l}(-1, \cdots, -1, p_l, \cdots, p_m)}{\mathfrak{l}(-1, \cdots, -1, p_l - \epsilon)}$$

are monotone by Proposition 90. Also, by Proposition 49, we have that

$$\Sigma\mathfrak{l}(-1,\cdots,-1,p_l+\epsilon)=\mathfrak{l}(-1,\cdots,-1,p_l+\epsilon+1)$$

and

$$\Sigma\mathfrak{l}(-1,\cdots,-1,p_l-\epsilon)=\mathfrak{l}(-1,\cdots,-1,p_l-\epsilon+1)$$

and therefore

$$\mathrm{P}\mathfrak{l}(-1,\cdots,-1,p_l-\epsilon)=\mathrm{P}\mathfrak{l}(-1,\cdots,-1,p_l+\epsilon)=\mathfrak{l}_l$$

Thus, by applying the typical squeezing argument from Proposition 43, we have that

$$\mathrm{P}\mathfrak{l}(-1,\cdots,-1,p_l,\cdots,p_m)=\mathfrak{l}_l$$

and therefore

$$\Sigma\mathfrak{l}(-1,\cdots,-1,p_l,\cdots,p_m)=\mathfrak{l}(-1,\cdots,-1,p_l+1,\cdots,p_m)$$

as claimed, which completes the proof of the first case. □

The notation used in Theorem 92 above might look a bit arcane. Together with Theorem 92, it describes an algorithm to calculate the growth order of the partial sum of the general growth order $\mathfrak{a}=\mathfrak{l}(p_0,\cdots,p_m)$:

1. If $p_0=p_1=\cdots=p_m=-1$, then the growth order of $\Sigma\mathfrak{a}$ is equal to $\mathfrak{l}_{m+1}$.
2. Otherwise, locate the first value of p_i which is not equal to -1. Say that this occurs at $i=l$, so that $p_l\neq-1$ and $p_i=-1$ for all $i<l$.
3. If $p_l<-1$, then $\Sigma\mathfrak{a}$ is the constant growth order $\mathbf{1}$.
4. If $p_l>-1$, then $\Sigma\mathfrak{a}$ is equal to $\mathfrak{n}\mathfrak{l}_1\cdots\mathfrak{l}_\ell\mathfrak{a}$.

This gives us, for instance, the following asymptotic formulae, among many others. Setting $p_0=-1$ and $p_1=p_2=-1/2$ gives the identity

$$\sum_{k=3}^{\infty}\frac{1}{k\sqrt{\log k\log\log k}}=\Theta\Big(\sqrt{\frac{\log n}{\log\log n}}\Big)$$

and setting $p_0=p_1=-1$, $p_2=1/2$ and $p_3=-1/3$ gives the identity

$$\sum_{k=16}^{\infty}\frac{\sqrt{\log\log k}}{k\log k\sqrt[3]{\log\log\log k}}=\Theta\Big(\frac{(\log\log k)^{3/2}}{\sqrt[3]{\log\log\log k}}\Big)$$

This general technique was already known to [8], who proved it using the technique of real induction. In Section 6.3, however, we derive summation formulae for an even broader class of growth orders that are not considered in that work, as well as additional moderateness and monotonicity results.

Now that we know how to calculate the growth orders of nested log sums in general, we are ready to prove that these are precisely the growth orders that appear in $\overline{\mathcal{M}}$.

Proposition 93. The elements of $\overline{\mathcal{M}}$ are precisely the growth orders taking the form $\mathrm{I}(p_0, \cdots, p_m)$, where $p_0, \cdots, p_m \in \mathbb{Z}$.

Proof. Define a "size" function on tuples of integers as follows:

$$\mathtt{size}(p_0, \cdots, p_m) = |p_0| + 2|p_1| + \cdots + 2^m|p_m|$$

Notice that this function has the following property: if $p_l > 0$, then

$$\mathtt{size}(0, \cdots, 0, p_l, \cdots, p_m) > \mathtt{size}(-1, \cdots, -1, p_l - 1, \cdots, p_m)$$

This fact will be important for proving our proposition using a modified type of induction on tuples of integers. Note that if the p_i weren't restricted to the integers, then this bound would *fail*. It does not hold for all tuples of real numbers.

Suppose we are given some $\mathrm{I}(p_0, \cdots, p_m)$, with $p_0, \cdots, p_m \in \mathbb{Z}$ not all equal to zero, which we wish to show is an element of $\overline{\mathcal{M}}$. Suppose that the sequence $p_0, \cdots, p_m$ begins with l zeroes, so that p_l is the first nonzero integer in the sequence, and

$$\mathrm{I}(p_0, \cdots, p_m) = \mathrm{I}(0, \cdots, 0, p_l, \cdots, p_m)$$

If $p_l > 0$, then we may write

$$\mathrm{I}(0, \cdots, 0, p_l, \cdots, p_m) = \Sigma\mathrm{I}(-1, \cdots, -1, p_l - 1, \cdots, p_m)$$

as per Theorem 92. On the other hand, if $p_l < 0$, we have

$$\mathrm{I}(0, \cdots, 0, p_l, \cdots, p_m) = \mathrm{R}\Sigma\mathrm{I}(-1, \cdots, -1, -p_l - 1, \cdots, -p_m)$$

However, notice that in the former case,

$$\mathtt{size}(-1, \cdots, -1, p_l - 1, \cdots, p_m) < \mathtt{size}(0, \cdots, 0, p_l, \cdots, p_m)$$

and in the latter case,

$$\mathtt{size}(-1, \cdots, -1, -p_l - 1, \cdots, -p_m) < \mathtt{size}(0, \cdots, 0, p_l, \cdots, p_m)$$

Thus, in either case, we can express $\mathrm{I}(p_0, \cdots, p_m)$ as the partial sum or the reciprocal of a partial sum of $\mathrm{I}(q_0, \cdots, q_m)$, where $\mathtt{size}(q_0, \cdots, q_m) < \mathtt{size}(p_0, \cdots, p_m)$.

Since $\mathtt{size}$ only takes nonnegative integer values, we have that $\mathtt{size}(q_0, \cdots, q_m) \leq \mathtt{size}(p_0, \cdots, p_m) - 1$. By repeatedly applying the above process, we decrease the size of the tuple of powers by at least 1 at each step, meaning that we must eventually reach a tuple whose size equals zero. The only such tuple is $(0, \cdots, 0)$. This means that $\mathrm{I}(p_0, \cdots, p_m)$ can be expressed in terms of

$\mathfrak{l}(0, \cdots, 0) = \mathbf{1}$ by repeatedly applying the Σ and $\mathrm{R}\Sigma$ operators. However, since $\overline{\mathcal{M}}$ contains $\mathbf{1}$ and is closed under Σ and $\mathrm{R}\Sigma$, it must contain $\mathfrak{l}(p_0, \cdots, p_m)$ for any $p_0, \cdots, p_m \in \mathbb{Z}$.

We have shown that $\overline{\mathcal{M}}$ contains all $\mathfrak{l}(p_0, \cdots, p_m)$ for $p_0, \cdots, p_m \in \mathbb{Z}$, and now we just need to show that it *only* contains growth orders of this form. Clearly $\mathcal{M}$ only contains growth orders of this form, since it consists of power functions $\mathfrak{n}^{p_0} = \mathfrak{l}(p_0)$ with $p_0 \in \mathbb{Z}$. However, Theorem 92 shows that if $\mathfrak{a}$ takes the form $\mathfrak{a} = \mathfrak{l}(p_0, \cdots, p_m)$ with $p_0, \cdots, p_m \in \mathbb{Z}$, then $\Sigma\mathfrak{a}$ and $\mathrm{R}\Sigma\mathfrak{a}$ also take this form. It follows inductively that all elements of $\mathcal{M}_i$ take this form for each $i \in \mathbb{N}$, and therefore all elements of $\overline{\mathcal{M}}$ take this form as well. □

This proposition demonstrates that the nested-logarithm growth orders with integer powers are, in some sense, fundamental to the study of partial sums of sequences: $\overline{\mathcal{M}}$ consists of precisely these growth orders, and every SR-closed set contains $\overline{\mathcal{M}}$.

6.3. Exponential extensions

We have seen how to use SR-regularity to construct chains of growth orders with certain favorable properties, such as moderateness, the monotone-quotient property, and SR-closure. Some of them, such as $\overline{\mathcal{M}}$, incidentally have other advantageous properties such as closure under products. Now we will derive a way of "extending" chains with these properties to obtain larger chains containing a large variety of growth orders and sharing the same favorable properties.

In some ways, what we will do is analogous to the idea of a field extension. Given a field such as $\mathbb{Q}$, some equations like $4x^2 - 1 = 0$ will have solutions, while other equations like $x^2 - 2 = 0$ will not have any solutions. We might want to construct a larger field containing $\mathbb{Q}$ but which also contains a solution to the equation $x^2 - 2 = 0$ without disrupting the field structure. This is the motivation behind the construction of the field $\mathbb{Q}(\sqrt{2})$. In our case, we will be considering equations of the form

$$\mathrm{P}\mathfrak{x} = \frac{\mathfrak{x}}{\Sigma\mathfrak{x}} = \mathfrak{a}$$

to be solved for $\mathfrak{x}$, where $\mathfrak{a} \in \mathcal{G}$ is some fixed growth order in a chain $\mathcal{G}$. If, for instance, $\mathcal{G} = \overline{\mathcal{M}}$, and $\mathfrak{a} = (\mathfrak{n}\mathfrak{l})^{-1}$, then $\mathfrak{x} = \mathfrak{n}^{-1}$ would be a solution to this equation. However, this equation does not have solutions for all values of $\mathfrak{a} \in \overline{M}$, for instance $\mathfrak{a} = \mathfrak{n}^{-1/2}$, $\mathfrak{a} = \mathbf{1}$, $\mathfrak{a} = \mathfrak{n}$, or $\mathfrak{a} = (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$.

A natural question is whether we can find extensions of $\overline{\mathcal{M}}$ in which these equations have solutions, and which share the favorable properties of moderateness, monotone quotients, SR-regularity, and SR-closure (and even closure under products). Clearly this is impossible for some values of $\mathfrak{a}$. For instance, the equation $\mathrm{P}\mathfrak{x} = \mathfrak{n}$ cannot have any solutions in any extension, because this would mean that $\Sigma\mathfrak{x} = \mathfrak{x}/\mathfrak{n} < \mathfrak{x}$, which is not the case for any growth order. To consider another example, the equation $\mathrm{P}\mathfrak{x} = \mathbf{1}$ does have some solutions, such as $\mathfrak{x} = [2^n]$, but none of these solutions are *moderate*, for we have proven that $\Sigma\mathfrak{x} \geq \mathfrak{n}\mathfrak{x}$ for all moderate growth orders $\mathfrak{x}$. (By similar reasoning, $\mathrm{P}\mathfrak{x} = \mathfrak{n}^{-1/2}$ also cannot have any moderate solutions.) This means that adjoining any solutions of this equation to $\overline{\mathcal{M}}$ would destroy its moderateness property. To continue the analogy with field extensions, this would be like trying to adjoin a solution to equations like $x = x + 1$ to $\mathbb{Q}$ - doing this would necessarily violate the field laws.

But what about the equation $\mathrm{P}\mathfrak{x} = (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$? We cannot rule out the existence of a solution to this equation for any of the above reasons, although we know that no such solution exists in $\overline{\mathcal{M}}$ because we have already completely classified its elements and their partial sums. Proposition 50 allows us to actually construct such a growth order, however:

$$\sum_{k=2}^{n} \frac{e^{\sqrt{\log k}}}{k\sqrt{\log k}} = \Theta\left(e^{\sqrt{\log n}}\right)$$

which tells us that $\Sigma\mathfrak{x} = \mathfrak{x}\mathfrak{n}\sqrt{\mathfrak{l}}$, and therefore $\mathrm{P}\mathfrak{x} = (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$, if $\mathfrak{x}$ is the growth order of $(e^{\sqrt{\log n}}/n\sqrt{\log n})$. Can we find some way of augmenting $\overline{\mathcal{M}}$ so that it contains this new growth order, while preserving all of its desirable properties?

This section is dedicated to proving that under certain conditions, a chain $\mathcal{G}$ of growth orders can be augmented by a solution $\mathfrak{x}$ to the equation $\mathrm{P}\mathfrak{x} = \mathfrak{b}$ without disrupting its favorable properties, such as trichotomy and closure under partial sums, products and quotients. Many of the results here are constructive, and the author has used them to implement an online growth-order calculator.

A comment about notation is in order before we proceed to proving various properties about exponential growth orders such as the above. When $\beta \in \mathfrak{b}$ is a particular sequence belonging to a certain growth order, we will *never* write $e^{\mathfrak{b}}$ and *always* instead write something like $[e^{\Sigma\beta}]$ which depends on the specific sequence β rather than the growth order $\mathfrak{b}$. This is because, as we saw in Section 3.3, exponentiation is *not a well-defined operation* on growth orders. That is, the growth order $[e^{\Sigma\beta}]$ can and often will vary when the chosen representative sequence $\beta \in \mathfrak{b}$ is varied. Writing $e^{\mathfrak{b}}$ would misleadingly suggest that exponentiation is well-defined, whereas writing $[e^{\Sigma\beta}]$ makes explicit the dependence on β.

The following proposition guarantees that moderateness is preserved when adding not only the above growth order, but also any growth order of the form $[e^{\Sigma\beta}]$, where β is any monotone decreasing sequence with "sufficiently fast decay" (e.g. with growth order $(\mathfrak{n}\sqrt{\mathfrak{l}})^{-1}$):

Proposition 94. If $\mathfrak{b} \le \mathfrak{n}^{-1}$ and $\beta \in \mathfrak{b}$, then $[e^{\Sigma\beta}]$ is a moderate growth order.

Proof. Let $\beta = (b_n) \in \mathfrak{b}$ be as stated in the proposition, and let $(c_n) = \Sigma e^{\beta}$. Further, let $m, n \in \mathbb{N}$ be such that $n \le m \le 2n$. Since $\mathfrak{b} \le \mathfrak{n}^{-1}$, there exists a constant C such that $b_n \le C/n$ for all $n \in \mathbb{N}$. Then we have that

$$\sum_{k=n+1}^{m} b_k \le \sum_{k=n+1}^{2n} \frac{C}{k} \le \sum_{k=n+1}^{2n} \frac{C}{n} = C$$

which means that

$$\exp\left(\sum_{k=1}^{n} b_k\right) \le \exp\left(\sum_{k=1}^{m} b_k\right) \le e^{C} \cdot \exp\left(\sum_{k=1}^{n} b_k\right)$$

and therefore we have that $e^{\Sigma\beta}$ has moderate growth order, by Proposition 7. $\square$

We have just shown that under certain conditions, moderateness is preserved when adding certain exponential growth orders. Now, in Proposition 97, we will state some conditions under which the monotone-quotient property is preserved, but we must first prove a few lemmas.

Proposition 95. Suppose $\mathfrak{a}/\mathfrak{b} > \mathbf{1}$ is monotone with $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ arbitrary. Then, for any constant $M > 0$, there exists $N \in \mathbb{N}$ such that $a_n/b_n > M$ for all $n \geq N$.

Proof. Let $(a_n) \in \mathfrak{a}$, $(b_n) \in \mathfrak{b}$ be as stated above, and let $(r_n) \in \mathfrak{a}/\mathfrak{b}$ be a monotone sequence. Since (r_n) is monotone and has a growth order strictly greater than $\mathbf{1}$, we have that for any constant > 0, there exists $N \in \mathbb{N}$ such that r_n exceeds the value of that constant for all $n \geq N$. Additionally, since $(r_n) \sim (a_n/b_n)$, we have that $a_n/b_n \geq Cr_n$ for all n, for some $C > 0$. We may therefore let $N \in \mathbb{N}$ be such that $r_n > M/C$ for all $n \geq N$, and consequently $a_n/b_n \geq M$ for all $n \geq N$, as claimed. $\square$

Proposition 96. Let $\mathcal{G}$ be a monotone, moderate and SR-regular set that is closed under quotients. Further let $\mathfrak{b} \in \mathcal{G}$ be $< \mathfrak{n}^{-1}$ and have the property that there exists no $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$. Then, if $\beta \in \mathfrak{b}$, for all $\mathfrak{a} \in \mathcal{G}$, the ratio $[e^{\Sigma\beta}]/\mathfrak{a}$ is monotone.

Proof. Let $\mathfrak{a} \in \mathcal{G}$ be arbitrary. Since $\mathcal{G}$ is SR-regular, we have that either $\mathfrak{a} = \Sigma\mathfrak{a}'$ or $\mathfrak{a} = (\Sigma\mathfrak{a}')^{-1}$ for some $\mathfrak{a}' \in \mathcal{G}$. If the latter is true, then we have that the ratio $[e^{\Sigma\mathfrak{b}}]/\mathfrak{a}$ is monotone, since $[e^{\Sigma\mathfrak{b}}]$ is monotone increasing and $1/\mathfrak{a}$ is monotone increasing.

Suppose instead that $\mathfrak{a} = \Sigma\mathfrak{a}'$. Since $\mathcal{G}$ is closed under quotients, we have that $\mathfrak{a}'/\Sigma\mathfrak{a}' = \mathfrak{a}'/\mathfrak{a}$ is in $\mathcal{G}$, and it therefore has a monotone ratio with $\mathfrak{b}$. We will consider two cases: either $\mathfrak{b}/(\mathfrak{a}'/\mathfrak{a})$ is monotone increasing and $> \mathbf{1}$, or it is monotone decreasing and $< \mathbf{1}$, for by hypothesis it cannot be $= \mathbf{1}$.

Suppose first that $\mathfrak{b}/(\mathfrak{a}'/\mathfrak{a})$ is monotone increasing and $> \mathbf{1}$. Let $\alpha' = (a'_n) \in \mathfrak{a}'$ be an arbitrary sequence of growth order $\mathfrak{a}'$, and let $\alpha = (a_n) = \Sigma\alpha' \in \mathfrak{a}$. By Proposition 95, and by moderateness, for any $M > 0$, the sequence $b_{n+1}/(a'_{n+1}/a_n)$ eventually exceeds M (for all $n \geq N$ with $N \in \mathbb{N}$). Notice, however, that if we fix some $M \geq 1$, we have

$$1 \leq \frac{1 + b_{n+1}}{1 + \frac{a'_{n+1}}{a_n}} \leq \frac{b_{n+1}}{a'_{n+1}/a_n}$$

for all $n \geq N$ by the mediant inequality [2], meaning that the quotient in the middle of the

[2] We saw this inequality in Section 4.3, but as a reminder, it states that for positive reals x, x', y, y' with $x/y < x'/y'$, we have

$$\frac{x}{y} < \frac{x + x'}{y + y'} < \frac{x'}{y'}$$

inequality exceeds 1 for all $n \geq N$. Since $e^h \geq 1 + h$ for all $h \in \mathbb{R}$, we have that

$$\frac{e^{b_{n+1}}}{1 + \frac{a'_{n+1}}{a_n}} \geq 1$$

for all $n \geq N$, or

$$\frac{e^{b_{n+1}}}{\frac{\sum_{k=1}^{n+1} a'_k}{\sum_{k=1}^{n} a'_k}} = \frac{e^{b_{n+1}}}{a_{n+1}/a_n} \geq 1$$

for all $n \geq N$. This expression is the ratio between consecutive terms of the sequence $e^{\Sigma\beta}/\alpha$. Since this ratio is greater than 1 for all $n \geq N$, it follows that the sequence is monotone increasing for all $n \geq N$, and therefore its growth order $[e^{\Sigma\beta}]/\mathfrak{a}$ is monotone increasing as desired.

Next consider the case in which $\mathfrak{b}/(\mathfrak{a}'/\mathfrak{a})$ is monotone decreasing, and define α' and α as before. By Proposition 95 and by moderateness once more, we have that for any $\epsilon > 0$, the sequence $(b_{n+1} + b_{n+1}^2)/(a'_{n+1}/a_n)$ is less than ϵ for all $n \geq N$, for some $N \in \mathbb{N}$. (We are using the fact that $(b_{n+1} + b_{n+1}^2)$ also has growth order $\mathfrak{b}$, which is a consequence of the hypothesis $\mathfrak{b} < \mathfrak{n}^{-1}$.) Using the mediant inequality again, if we fix some positive $\epsilon < 1$, this means that

$$\frac{b_{n+1} + b_{n+1}^2}{a'_{n+1}/a_n} \leq \frac{1 + b_{n+1} + b_{n+1}^2}{1 + \frac{a'_{n+1}}{a_n}} \leq 1$$

for all $n \geq N$. Now, notice that $e^x \leq 1 + x + x^2$ for all sufficiently small x, meaning that since b_{n+1} tends to zero, we have that $e^{b_{n+1}} \leq 1 + b_{n+1} + b_{n+1}^2$ for all $n \geq N'$, for some $N' \in \mathbb{N}$. This means that

$$\frac{e^{b_{n+1}}}{1 + \frac{a'_{n+1}}{a_n}} \leq \frac{1 + b_{n+1} + b_{n+1}^2}{1 + \frac{a'_{n+1}}{a_n}} \leq 1$$

for all $n \geq \max(N, N')$. But this means that

$$\frac{e^{b_{n+1}}}{\frac{\sum_{k=1}^{n+1} a'_k}{\sum_{k=1}^{n} a'_k}} = \frac{e^{b_{n+1}}}{a_{n+1}/a_n} \leq 1$$

and the LHS is the ratio between consecutive terms of the sequence $e^{\Sigma\beta}/\alpha$. Since these ratios are less than 1 for all $n \geq \max(N, N')$, the sequence must be monotone decreasing for all n above this threshold, and therefore the growth order $[e^{\Sigma\beta}]/\mathfrak{a}$ is monotone decreasing as claimed, completing our proof. □

We have the following corollary of Proposition 96, which follows by noticing that for any scalar $p > 0$, the representative sequence $\beta \in \mathfrak{b}$ can be replaced by $p\beta$, causing $[e^{\Sigma\beta}]$ to become $[e^{p\Sigma\beta}]$, or equivalently $[e^{\Sigma\beta}]^p$.

Proposition 97. Let $\mathcal{G}$ be a monotone, moderate and SR-regular set that is closed under quotients. Further let $\mathfrak{b} \in \mathcal{G}$ be $< \mathfrak{n}^{-1}$ and have the property that there exists no $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$, and let $\beta \in \mathfrak{b}$. For all $\mathfrak{a} \in \mathcal{G}$, either $[e^{\Sigma\beta}]^p/\mathfrak{a}$ is monotone increasing for all $p > 0$, or monotone decreasing for all $p > 0$.

The following proposition will allow us to claim SR-closure when we extend certain well-behaved collections of growth orders $\mathcal{G}$ by adding the products of its preexisting elements with growth orders taking the form $[e^{\Sigma\beta}]^p$. In particular, it provides a simple algorithm for explicitly computing the partial sums of growth orders taking the form $\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$, where $\mathfrak{g}, \mathfrak{b} \in \mathcal{G}$.

Proposition 98. Let $\mathcal{G}$ be a moderate monotone-quotient SR-regular and SR-closed set that is closed under quotients. Let $\mathfrak{b}, \mathfrak{g} \in \mathcal{G}$ and let $\beta \in \mathfrak{b}$ be monotone decreasing such that $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, and such that there exists no growth order $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$.
Then if

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p)$$

we have that

1. If $\mathfrak{g} = \Sigma\mathfrak{g}'$ and $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$, then $\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ if $p > 0$ and $\mathfrak{a} = \mathbf{1}$ if $p < 0$.
2. If $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$ and $\mathfrak{g}'/\mathfrak{g}^{-1} < \mathfrak{b}$, then $\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ if $p > 0$ and $\mathfrak{a} = \mathbf{1}$ if $p < 0$.
3. If $\mathfrak{g} = \Sigma\mathfrak{g}'$ and $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$, then $\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b}) \cdot [e^{\Sigma\beta}]^p$.
4. If $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$ and $\mathfrak{g}'/\mathfrak{g}^{-1} > \mathfrak{b}$, then $\mathfrak{a} = \mathbf{1}$.

Proof. First let us consider case (1) with $p > 0$. In the proofs of Proposition 97 and Proposition 98, we showed that if $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$, then $\mathfrak{g} < [e^{\Sigma\beta}]^q$ for all $q > 0$. This means that

$$\mathbf{1} \le \mathfrak{g} \le [e^{\Sigma\beta}]^p$$

and so, multiplying across by $\mathfrak{b}[e^{\Sigma\beta}]^p$, we have that

$$\mathfrak{b} \cdot [e^{\Sigma\beta}]^p \le \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \le \mathfrak{b} \cdot [e^{\Sigma\beta}]^{2p}$$

where there is a monotone ratio between any two of these growth orders. However, by Proposition 49, replacing $\mathfrak{a}$ with $\mathfrak{b}[e^{\Sigma\beta}]$ in the statement of the proposition, we find that

$$\mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^{2p}) = \mathfrak{b}$$

and therefore, using Proposition 43 and a squeezing argument, we have that

$$\mathrm{P}(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{b}$$

thus

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

as claimed. If, on the other hand, $p < 0$, then we have that

$$\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \le \mathfrak{b} \cdot [e^{\Sigma\beta}]^{p/2}$$

and the latter growth order has convergent partial sums, by Proposition 49, so the former must as well, meaning that $\mathfrak{a} = \mathbf{1}$. Thus follows the claim for the case of $p < 0$.

Now let us consider case (2) - the proof will be very similar. Suppose first that $p > 0$. This time, we have that since $\mathfrak{g}'/\mathfrak{g}^{-1} < \mathfrak{b}$, it follows that $\mathfrak{g} > [e^{\Sigma\beta}]^q$ for all $q < 0$. This means that

$$\mathfrak{b} \cdot [e^{\Sigma\beta}]^{p/2} \leq \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$$

where any two of these growth orders has a monotone ratio. By Proposition 49, we have that

$$\mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^{p/2}) = \mathrm{P}(\mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{b}$$

and therefore by the same squeezing argument, we have

$$\mathrm{P}(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{b}$$

and thus

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

as claimed. Once again, if $p < 0$, we have that

$$\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$$

and the latter has convergent sums, meaning that the former does as well, so $\mathfrak{a} = \mathbf{1}$ and the second part of case (2) follows.

Next, we shall prove the claim for case (3). Because $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$, we have by Proposition 45 and the fact that $\Sigma\mathfrak{b} > \mathbf{1}$ that $\Sigma(\mathfrak{g}'/\mathfrak{g}) > \Sigma\mathfrak{b}$, or $\mathfrak{l} \circ \mathfrak{g} > \Sigma\mathfrak{b}$. But this means that $\mathfrak{g} > [e^{\Sigma\beta}]^q$ for all $q > 0$. This implies that

$$\Sigma(\mathfrak{g} \cdot \mathfrak{b}) \geq \Sigma([e^{\Sigma\beta}]^q \cdot \mathfrak{b}) = [e^{\Sigma\beta}]^q$$

for all $q > 0$, with a monotone quotient. From this inequality, we have that

$$\frac{\mathfrak{g} \cdot \mathfrak{b}}{\sqrt{\Sigma(\mathfrak{g} \cdot \mathfrak{b})}} \leq \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p \leq \mathfrak{g} \cdot \mathfrak{b} \cdot \Sigma(\mathfrak{g} \cdot \mathfrak{b})$$

with monotone quotients. However, the growth orders on the left and the right both have a sigma ratio equal to $(\mathfrak{g} \cdot \mathfrak{b})/\Sigma(\mathfrak{g} \cdot \mathfrak{b})$ by Proposition 49 and moderateness. Thus, by Proposition 43, we have that $\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$ has the same sigma ratio, meaning that

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \Sigma(\mathfrak{g} \cdot \mathfrak{b}) \cdot [e^{\Sigma\beta}]^p$$

completing our proof of case (3).

Finally, we shall turn to case (4). Using the same line of reasoning as case (3), since $\mathfrak{g}'/\mathfrak{g}^{-1} > \mathfrak{b}$, we have again that $\Sigma(\mathfrak{g}'/\mathfrak{g}^{-1}) = \mathfrak{l} \circ \mathfrak{g}^{-1} > \Sigma\mathfrak{b}$, and therefore $\mathfrak{g}^{-1} > [e^{\Sigma\beta}]^q$ for all $q > 0$, or $\mathfrak{g} < [e^{\Sigma\beta}]^q$ for all $q < 0$. This means that

$$\Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) \leq \Sigma(\mathfrak{b} \cdot [e^{\Sigma\beta}]^{p+q}) = \mathbf{1}$$

for sufficiently large negative $q < 0$. Therefore, since $\mathfrak{a}$ can be no smaller than $\mathbf{1}$, we have that

$$\mathfrak{a} = \Sigma(\mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p) = \mathbf{1}$$

as claimed. □

In addition to bringing us closer to our goal of being able to enlarge chains while preserving their desirable properties, this proposition allows us to deduce a smattering of new summation identities. For instance, applying the first case with the values

$$\begin{aligned} \mathfrak{g}' &= (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1} \\ \mathfrak{g} &= \sqrt{\mathfrak{l}} \\ \mathfrak{g}'/\mathfrak{g} &= (\mathfrak{n}\mathfrak{l})^{-1} \\ \mathfrak{b} &= (\mathfrak{n}\sqrt{\mathfrak{l}})^{-1} \end{aligned}$$

yields the formula

$$\sum_{k=2}^{n} \frac{e^{\sqrt{\log k}}}{k} = \Theta\big(\sqrt{\log n} \cdot e^{\sqrt{\log n}}\big)$$

whereas, for instance, applying the fourth case to the values

$$\begin{aligned} \mathfrak{g}' &= (\mathfrak{n}\mathfrak{l})^{-1} \\ \mathfrak{g} &= \mathfrak{l}_2^{-1} \\ \mathfrak{g}'/\mathfrak{g}^{-1} &= (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2)^{-1} \\ \mathfrak{b} &= (\mathfrak{n}\mathfrak{l}\mathfrak{l}_2\mathfrak{l}_3^{2/3})^{-1} \end{aligned}$$

yields the formula

$$\sum_{k=100}^{n} \frac{e^{\sqrt[3]{\log\log\log n}}}{n \log n (\log\log n)^2 (\log\log\log n)^{2/3}} = \Theta(1)$$

Finally, we are ready to prove a proposition showing that SR-regularity is preserved when extending certain chains $\mathcal{G}$ of growth orders.

Proposition 99. Let $\mathcal{G}$ be a moderate monotone-quotient SR-regular and SR-closed set that is closed under quotients. Further let $\mathfrak{b}, \mathfrak{g} \in \mathcal{G}$ and let $\beta \in \mathfrak{b}$ be monotone decreasing such that $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, and such that there exists no growth order $\mathfrak{c} \in \mathcal{G}$ with $\mathfrak{c}/\Sigma\mathfrak{c} = \mathfrak{b}$. Then, for any given $p \in \mathbb{R}$, there exists a growth order $\mathfrak{a} \in \mathcal{G}$ such that either

$$\Sigma\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

or

$$(\Sigma\mathfrak{a})^{-1} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$$

Proof. By the SR-regularity of $\mathcal{G}$, there exists $\mathfrak{g}' \in \mathcal{G}$ such that either $\mathfrak{g} = \Sigma\mathfrak{g}'$ or $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$. If $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$ or $\mathfrak{g}'/\mathfrak{g}^{-1} < \mathfrak{b}$ respectively, then we have that setting

$$\mathfrak{a} = \mathfrak{g} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$$

gives $\Sigma\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ for $p > 0$, by Proposition 98. If, on the other hand, $p < 0$, we may let

$$\mathfrak{a} = \mathfrak{g}^{-1} \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^{-p}$$

so that we have $(\Sigma\mathfrak{a})^{-1} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ by Proposition 98. Therefore, the claim holds for the case in which $\mathfrak{g}'/\mathfrak{g} < \mathfrak{b}$ or $\mathfrak{g}'/\mathfrak{g}^{-1} < \mathfrak{b}$, and now we need only consider the case in which $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$ or $\mathfrak{g}'/\mathfrak{g}^{-1} > \mathfrak{b}$. (It is not possible for $\mathfrak{g}'/\mathfrak{g}$ to *equal* $\mathfrak{b}$, by hypothesis.)

Consider now the case in which $\mathfrak{g} = \Sigma\mathfrak{g}'$ and $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$. We may equivalently write this inequality as $\mathfrak{g}'/\mathfrak{b} > \mathfrak{g}$. Since $\mathfrak{g} \geq \mathbf{1}$, we have that $\mathfrak{g}'/\mathfrak{b} > \mathbf{1}$, and therefore by SR-regularity of $\mathcal{G}$, we have that $\mathfrak{g}'/\mathfrak{b} = \Sigma\mathfrak{g}''$ for some $\mathfrak{g}'' \in \mathcal{G}$. Since $\Sigma\mathfrak{g}'' = \mathfrak{g}'/\mathfrak{b} > \mathfrak{g} = \Sigma\mathfrak{g}'$, we have that $\mathfrak{g}'' > \mathfrak{g}'$ (for the two growth orders must be comparable, and if $\mathfrak{g}'' \leq \mathfrak{g}'$ were true, it would follow that $\Sigma\mathfrak{g}'' \leq \Sigma\mathfrak{g}'$, which is not the case). Now, by Proposition 43, we have that $\mathrm{P}\mathfrak{g}'' \geq \mathrm{P}\mathfrak{g}'$, implying that

$$\frac{\mathfrak{g}''}{\mathfrak{g}'/\mathfrak{b}} = \frac{\mathfrak{g}''}{\Sigma\mathfrak{g}''} \geq \frac{\mathfrak{g}'}{\Sigma\mathfrak{g}'} = \frac{\mathfrak{g}'}{\mathfrak{g}} > \mathfrak{b}$$

Therefore, by Proposition 98, we have that if we let

$$\mathfrak{a} = (\mathfrak{g}'/\mathfrak{b}) \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p = \mathfrak{g}' \cdot [e^{\Sigma\beta}]^p$$

it would follow that $\Sigma\mathfrak{a} = \mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ for any $p \in \mathbb{R}$, proving the proposition for this case.

For the final case in which $\mathfrak{g} = (\Sigma\mathfrak{g}')^{-1}$ and $\mathfrak{g}'/\mathfrak{g} > \mathfrak{b}$, we may consider the growth order $\mathfrak{g}^{-1} \cdot [e^{\Sigma\beta}]^{-p}$ and notice that it falls under the previous case, for which the proposition was just proven. Thus, the growth order $\mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ has a preimage under Σ or $\mathrm{R}\Sigma$ in all cases, and the proposition is proven. □

Theorem 100. *Let $\mathcal{G}$ be a set of growth orders that is*
(1) moderate,
(2) monotone-quotient,
(3) SR-regular,
(4) SR-closed,
(5) and closed under quotients.
Let $\mathfrak{b} \in \mathcal{G}$ be a growth order with $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, and let $\beta \in \mathfrak{b}$ be monotone decreasing. Then, if we denote by $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ the set

$$\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A = \{\mathfrak{g} \cdot [e^{\Sigma\beta}]^p \ : \ \mathfrak{g} \in \mathcal{G},\ p \in A\}$$

where A is an additive subgroup of $\mathbb{R}$, we have that $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ also satisfies properties (1) through (5).

Proof. Let $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ be as described above. The fact that $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ satisfies (1) follows from Proposition 94. The fact that it satisfies (2) follows from Proposition 97. The fact that it is SR-regular follows from Proposition 99.

To see why $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ is SR-closed, notice that an arbitrary element $\mathfrak{g} \cdot [e^{\Sigma\beta}]^p$ can be rewritten in the form $(\mathfrak{g}/\mathfrak{b}) \cdot \mathfrak{b} \cdot [e^{\Sigma\beta}]^p$. It follows from Proposition 98 that the partial sum of this growth order is equal to either $(\mathfrak{g}/\mathfrak{b}) \cdot [e^{\Sigma\beta}]^p$ or $(\Sigma\mathfrak{g}) \cdot [e^{\Sigma\beta}]^p$ or $\mathbf{1}$. Clearly all three of these growth orders belong to $\mathcal{G}[e_A^{\Sigma\mathfrak{b}}]$, since $\mathcal{G}$ is closed under quotients and SR-closed. Thus follows property (4).

Finally, the quotient of two arbitrary elements $\mathfrak{g}_1 \cdot [e^{\Sigma\beta}]^p$ and $\mathfrak{g}_2 \cdot [e^{\Sigma\beta}]^q$ of $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ can be written as

$$\frac{\mathfrak{g}_1 \cdot [e^{\Sigma\beta}]^p}{\mathfrak{g}_2 \cdot [e^{\Sigma\beta}]^q} = (\mathfrak{g}_1/\mathfrak{g}_2) \cdot [e^{\Sigma\beta}]^{p-q}$$

which is an element of $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ because A is an additive subgroup of $\mathbb{R}$, and because $\mathfrak{g}_1/\mathfrak{g}_2 \in \mathcal{G}$ since $\mathcal{G}$ is closed under quotients. Hence, $\mathcal{G}[e^{\Sigma\mathfrak{b}}]_A$ satisfies (5) as well, and the claim is proven. □

Theorem 101. *Let $\mathcal{G}$ be a moderate, monotone-quotient, SR-regular, and SR-closed set of growth orders that is also closed under quotients. Then there exists a set $\mathcal{G}_{\exp} \supset \mathcal{G}$, which we shall call the* ***exponential closure*** *of $\mathcal{G}$, that satisfies each of these five properties, but also has the following property: for any $\mathfrak{b} \in \mathcal{G}_{\exp}$ with $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, there exists $\mathfrak{a} \in \mathcal{G}_{\exp}$ such that $\mathrm{P}\mathfrak{a} = \mathfrak{b}$.*

Proof. Let $\mathcal{P} \subset 2^{\mathcal{S}(\mathbb{R}^+)/\sim}$ be defined as the set of moderate, monotone-quotient, SR-regular and SR-closed supersets of $\mathcal{G}$ that are also closed under quotients. Define a partial ordering on $\mathcal{P}$ by letting $\mathcal{G}_1 \leq \mathcal{G}_2$ iff $\mathcal{G}_1 = \mathcal{G}_2$ or if there exists some $\beta \in \mathfrak{b} \in \mathcal{G}_1$ such that $\mathcal{G}_1[e^{\Sigma\beta}]_A \subset \mathcal{G}_2$ for which the exponential extension $\mathcal{G}_1[e^{\Sigma\beta}]_A$ is defined. Notice that for any chain in this partial ordering, taking the union of all elements in that chain results in an element of $\mathcal{P}$ that is greater than or equal to every element of that chain. Thus, every chain of $\mathcal{P}$ has an upper bound.

Now we may apply Zorn's Lemma and conclude that $\mathcal{P}$ necessarily has a maximal element, namely a set $\mathcal{G}_{\exp} \supset \mathcal{G}$ such that *there exists no $\beta \in \mathfrak{b} \in \mathcal{G}_{\exp}$* for which the extension $\mathcal{G}_{\exp}[e^{\Sigma\mathfrak{b}}]_A$ is defined, for if it were defined, it would be a proper exponential extension of $\mathcal{G}_{\exp}$ satisfying each of the five desired properties. Hence, if $\mathfrak{b} \in \mathcal{G}_{\exp}$ is such that $\mathfrak{b} < \mathfrak{n}^{-1}$ and $\Sigma\mathfrak{b} > \mathbf{1}$, it cannot be the case that there is no $\mathfrak{a} \in \mathcal{G}_{\exp}$ with $\mathrm{P} = \mathfrak{b}$, for if no such $\mathfrak{a}$ existed, the extension $\mathcal{G}_{\exp}[e^{\Sigma\mathfrak{b}}]_A$ could be constructed. Hence, for every such $\mathfrak{b} \in \mathcal{G}_{\exp}$, there must exist $\mathfrak{a} \in \mathcal{G}_{\exp}$ such that $\mathrm{P}\mathfrak{a} = \mathfrak{b}$. □

It may seem at first like the study of these strange exponential growth orders is esoteric and without real applications. However, growth orders of the forms described above often make surprising appearances in analytic number theory. For instance, the family of growth orders

$$\left[n \cdot \exp\left(-c'' \cdot \frac{\ln n \cdot \ln\ln\ln n}{\ln\ln n}\right)\right]$$

makes an appearance in [6] as an upper bound for the number of Carmichael Numbers below n, and this growth order is a special case of $[e^{\Sigma\beta}]$ in which $\mathfrak{b} = [\beta] = \mathfrak{l}_3/\mathfrak{n}\mathfrak{l}_2$. If, say, we wanted to find a *lower bound* for the nth largest Carmichael number, we might concern ourself with finding the compositional inverse of this growth order, as described in Section 5.4.

As another example, the following growth order arises in [5] during the study of general number field sieves for integer factorization algorithms:

$$\left[\exp\left(c'\sqrt[3]{\ln n \cdot \ln^2 \ln n}\right)\right]$$

which is a special case of $[e^{\Sigma\beta}]$ in which $\mathfrak{b} = [\beta] = \mathfrak{n}^{-1}(\mathfrak{l}_2/\mathfrak{l})^{2/3}$.

6.4. Wrap-up and future directions

In previous chapters, we have seen some of the ways in which the most general collection of growth orders $\mathcal{S}(\mathbb{R}^+)/\sim$ is poorly behaved and unintuitive. In particular, in Section 3.4 we observed for the first time that the natural partial ordering on growth orders fails (and fails *spectacularly*) to be a total ordering. We saw in Proposition 29 that this lack of total ordering can be salvaged somewhat by uncovering a lattice structure on $\mathcal{S}(\mathbb{R}^+)/\sim$, but even this ordering is somewhat messy. For instance, in Proposition 30 we saw that not only is it not a complete lattice, but it is a lattice in which *no increasing sequence* has a least upper bound.

Thus, one of our main goals has been to fashion a more restricted set of growth orders that behaves more favorably. Much of the earlier chapters of this thesis is dedicated to figuring out what exactly constitutes "favorable behavior" in growth orders. In Section 2.3 we introduced moderate growth, and in Section 5.1 we saw that this property guarantees for us the well-definedness of composition. In Section 4.2 we introduced monotonicity of growth orders and saw how it gives limited guarantees of comparability.

In Section 6.2, we brought together many of the propositions proven about the elementary operations on growth orders, including multiplication, partial sums, and composition, to construct a "small" set of well-behaved growth orders $\overline{\mathcal{M}}$ that is closed under several of the important growth order operations. Finally, in Section 6.3, we saw how this set could be used as a starting point from which to build up more interesting growth orders without destroying their favorable properties. Many intermediate results, such as Theorem 92 and Proposition 98, have been constructive, giving rise to algorithmic procedures for computing partial sums of growth orders. These algorithms have culminated in the design of the "asymptotic calculator" given in [3].

Because so much overhead was necessary to construct exponential extensions in the first place, the author suspects that much is left to be discovered about these structures and the growth orders comprising them. For instance, although the composition operation was helpful in Section 6.2 for computing partial sums of products of nested logarithms, the author still knows little about how the elements of exponential extensions behave under composition (besides the fact that they *can be* composed, due to their moderateness). Furthermore, Proposition 74 hints that there may be deep, non-obvious connections between exotic exponential growth orders like $[e^{\sqrt{\log n}}]$ and the composition operation.

To that end, the author would like to propose the following questions for further investigation. When $\mathcal{G}$ is a known moderate, monotone-quotient SR-closed chain and $\mathcal{G}[e^{\Sigma\beta}]_A$ an exponential extension thereof:

- What are sufficient conditions for $\mathcal{G}[e^{\Sigma\beta}]_A$ to be closed under composition?
- Can an algorithm be given for computing the compositional inverses of elements of $\mathcal{G}[e^{\Sigma\beta}]_A$?
- Can an algorithm be given for deciding whether one element of $\mathcal{G}[e^{\Sigma\beta}]_A$ absorbs another element?

There are also several other potential operations on growth orders that the author has spent some time exploring independently. Among these are operations for expressing the growth orders of sequences given by *recurrence relations*, which are particularly relevant in measuring the performance of algorithms in theoretical computer science. These include recurrences such as the "divide and conquer" recurrence

$$T(n) = AT(n/B) + a_n$$

and recurrences that count the number of iterations of a function required to reach a base case, such as those taking the form

$$T(n) = T(a_n) + 1$$

The author has been able to prove that, under certain conditions, these are well-defined operations in the sense that the growth order of the sequence $(T(n))$ depends only on the growth order of (a_n). The author also suspects that understanding these recurrences, specifically the latter, are indispensable when it comes to studying absorption. However, this theory has not been developed sufficiently to be included in the thesis.

In summary, the author feels that there is still much to be understood about the algebraic structure underlying asymptotic growth orders. As someone who often manipulates asymptotic estimates while solving problems, whether it be to measure the performance of an algorithm, estimate a combinatorial or number-theoretic sequence, or analyze the limiting behavior of a function, this investigation has given the author a deeper appreciation of the nuances underlying asymptotic analysis. Hopefully some part of this thesis will inspire someone else to think a little more deeply about asymptotic growth orders as well, as algebraic objects in their own right.

A. Lists for quick reference

A.1. List of counterexamples

The following is a list of pathological (but illuminating!) counterexamples appearing throughout the document:

1. Two incomparable growth orders:
   section 3.4, page 20
2. An uncountable antichain of incomparable growth orders:
   section 3.4, page 21
3. An increasing sequence of growth orders without a least upper bound:
   section 3.5, page 23
4. Two monotone yet incomparable growth orders:
   section 4.2, page 30
5. A sequence whose arithmetic subsequences have different growth order, in particular with (a_{2n}) growing faster than (a_n):
   section 2.3, page 13
6. A sequence whose translations have a different growth order, in particular with (a_{n+1}) growing faster than (a_n):
   section 2.3, page 13
7. A sequence that is bounded by polynomials yet immoderate:
   section 2.3, page 10
8. A sequence that is moderate yet incomparable to $\mathbf{1}$:
   section 4.2, page 31
9. Sequences demonstrating that subtraction of growth orders is ill-defined:
   section 3.3, page 18
10. Sequences demonstrating that exponentiation of growth orders is ill-defined:
    section 3.3, page 19
11. Two distinct incomparable growth orders with the same partial sum:
    section 4.1, page 27
12. Two distinct comparable growth orders with the same partial sum:
    section 4.1, page 27

13. Two growth orders $\mathfrak{a} < \mathfrak{b}$ such that $\mathfrak{a}/\Sigma\mathfrak{a} > \mathfrak{b}/\Sigma\mathfrak{b}$:
section 4.3, page 36

14. A growth order which decays faster than each growth order of the form $(\mathfrak{n}\mathfrak{l}_1 \cdots \mathfrak{l}_m)^{-1}$, yet still has divergent partial sums:
section 4.5, page 42

15. Growth orders $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ such that $\mathfrak{a} \geq 1$ and $\mathfrak{b} \leq \mathfrak{c}$ but $\mathfrak{a} \circ \mathfrak{b} \perp \mathfrak{a} \circ \mathfrak{c}$:
section 5.2, page 49

A.2. List of operations

Over the course of this document, we have defined several operations on growth orders, some of which require certain properties of their operands in order to be well-defined. The following is a list of operations, as well as the criteria that they require:

Name	Notation	Definition	Criteria
Addition	$\mathfrak{a} + \mathfrak{b}$	Growth order of $(a_n + b_n)$	All growth orders $\mathfrak{a}, \mathfrak{b}$
Multiplication	$\mathfrak{a} \cdot \mathfrak{b}$	Growth order of $(a_n \cdot b_n)$	All growth orders $\mathfrak{a}, \mathfrak{b}$
Partial summation	$\Sigma\mathfrak{a}$	Growth order of $(\Sigma_{k=1}^{n} a_k)$	All growth orders $\mathfrak{a}$
Partial summation ratio	$\mathrm{P}\mathfrak{a}$	$\mathfrak{a}/\Sigma\mathfrak{a}$	All growth orders $\mathfrak{a}$
Composition	$\mathfrak{a} \circ \mathfrak{b}$	Growth order of (a_{b_n})	$\mathfrak{a}$ moderate, $\mathfrak{b} \geq \mathbf{1}$
Inversion	$\mathrm{inv}(\mathfrak{a})$	Growth order of (b_n) such that $(a_{b_n}) \sim (b_{a_n}) \in \mathfrak{n}$	$\mathfrak{a}$ moderate, $\mathfrak{n}^p/\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{n}^q$ monotone increasing for some $p, q > 0$

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