An Algebraist's Introduction to Multiresolution Analysis

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This thesis is dedicated to Nic, my old, dear friend, who since the very beginning inspires me.

Abstract

Multiresolution analysis is in many ways a successor to Fourier analysis and continues to see many novel and fascinating applications. While the development of abstract Fourier analysis made those techniques accessible to algebraists, the abstract formulations of multiresolution analyses and wavelets have been more obscure. We will detail approaches to constructing wavelets in $L^2(G)$ for locally-compact abelian groups G, both with and without a discrete subgroup to serve as a lattice, as well as provide a constructive, module-theoretic description for multiresolution analyses in greater generality.

Contents

0 Introduction

Fourier analysis is remarkable in the breadth of its impact as a theory, spanning engineering applications to pure number theory. In many physical applications, however, Fourier analysis has been superseded by multiresolution analysis due in part to the former's difficulties handling of discontinuities in the discrete case. However, though the wonders of Fourier analysis on locally-compact abelian groups are very canonical, there is less of a consistent interpretation of multiresolution analysis in an algebraic sense. One productive approach has been to, following multiresolution analysis's dependence on Fourier theory, develop wavelets in L^2 complex-valued function spaces of locally-compact abelian groups. Alternately, the chain of subspaces of $L^2(\mathbb{R})$, for instance, that formally constitute a multiresolution analysis themselves have the structure of modules over a ring of transformations. Although we do not manage to unify these theories, we will develop both in parallel.

This thesis is intended for a dual audience and should be accessible to anyone with an undergraduate background in either commutative algebra or harmonic analysis. For specialists or as a reminder, the first section will review some fundamental concepts from algebra and category theory, and then the preliminaries of harmonic analysis, including abstract Fourier theory. We will then revisit the classical approach to multiresolution analysis on $L^2(\mathbb{R})$, and will conclude with a description of the Haar wavelet.

The second section will provide an algebraic justification for our generalizations of components of the multiresolution analysis apparatus that we described in Section 1.2. In particular, we will outline the theories of Pontryagin duality using category theory, which will justify a more general definition of a lattice and its adjoint. Second, we will discuss the interrelation between the translation and dilation operators and how a discrete subgroup of a locally-compact abelian group can take the place of the translation lattice $\mathbb{Z}$ in traditional multiresolution analysis over $\mathbb{R}$. Third, we will generalize notions of expansiveness, a quality which is preferred [but not required, see Spe03] of an automorphism from which the dilation operator is derived. Following the work of J. and R. Benedetto, we will then describe a method of constructing a lattice in a group without a discrete subgroup to serve as one.

In the third section, we take a step back and, without specifying the actions of a translation or a dilation operator, describe how such operators form a group of transformations that, acting on a symbolic "wavelet," yield a module structure. We show that this accurately reproduces the Haar multiresolution analysis on $L^2(\mathbb{R})$, and that Benedetto's construction of the translation operator also accurately reproduces the usual one, and begin such a construction on $L^2(\mathbb{F}_p((t)))$.

In section four, we complete the construction on $L^2(\mathbb{F}_p((t)))$ and describe a wavelet set in this case. We then give a less-strict criterion for the existence of wavelet sets from the module theory perspective, concluding with the construction of the Haar wavelet from its scaling equation using this technique.

1 Preliminaries

We begin by reviewing the algebraic constructions we will need: groups, rings, modules, and group rings. In order to justify the consistence of our work with classical harmonic analysis, we will require a small amount of category theory, so we will review functors and exactness. In Section 1.2, we will briefly discuss some topics we will need from real analysis, such as Lebesgue integration and the Haar measure, as well as Pontryagin's Fourier theory on locally-compact abelian groups. In Section 1.3, we motivate the classical multiresolution analysis on $L^2(\mathbb{R})$ and describe the Haar wavelet, which will reappear throughout the text.

1.1 Algebra

Because we intend for this thesis to be accessible to analysts, we will give some basic definitions of algebraic structures. We will also give a brief description of category-theoretic ideas that pertain to our work. No deep results from category theory are used; rather, we find the language of category theory to be helpful in justifying the consistency of algebraic multiresolution analysis with the classical formulation. Our references are to standard undergraduate- or graduate-level textbooks that we would recommend if one wishes to dive deeper. In particular, for category theory, [Rie14] is an excellent starting point for what can be a difficult and unwieldy topic.

A *group* is a set with the additional structure of an associative operation, generally denoted by multiplication. Each group has an identity element 1 such that $1x = x$, and each element x has an inverse such that $xx^{-1} = 1$. A group is *abelian* if $xy = yx$ for every x, y in the group [DF04, Section 1.1]. A *topological group* is a group enriched with a topology such that the group's inverse operation is a continuous function on the group, and the multiplication operation is a continuous function from the product group to the group. A locally compact abelian group (**LCA** group) is a topological group that is locally compact as a topological space, i.e. every point has a compact neighborhood [Rud90, A4].

A *ring* is an abelian group, whose operation we denote by $+$, with a second associative operation called multiplication that is distributive over addition, i.e. $a(b + c) = ab + bc$. A ring has an additive identity 0 and, while some authors do not require it, we will require the presence of a multiplicative identity 1. If $ab = ba$ for every a, b in the ring, we call it a *commutative ring* [DF04, Section 7.1]. In general, ring elements don't have multiplicative inverses – if there exists a^{-1} for each a in a commutative ring such that $aa^{-1} = 1$, then we call the ring a *field*.

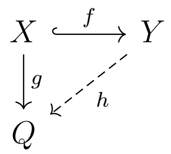
Analogously to a vector space over a field, we can define an $R-$*module* over a ring R. If R is noncommutative then there are distinct left and right $R-$modules; otherwise, they coincide. Module elements may be added to each other and multiplied by elements of R obeying a distributive property, i.e. for $r \in R$ and elements x, y of a module M, $r(x + y) =$

$rx + ry \in M$. Modules may be generated by a set of elements, in which case the underlying set of the module is the scalar product of each ring element with each generator. An $R-$module generated by a single element x is called *cyclic* and denoted xR [DF04, Section 10.1]. Modules without a generating set exist (such as $\mathbb{Q}$ as a $\mathbb{Z}-$module), but our work does not concern them. A *group ring* is a ring RG formed from a group G and a ring R by symbolically concatenating elements of the group with elements of the ring, giving it the structure of an $R-$module with basis G [DF04, Section 7.2].

Each of these structures admits substructures (subgroups, ideals, and submodules) and *homomorphisms*, which are maps obeying a multiplicative rule, i.e. for groups $\varphi(ab) = \varphi(a)\varphi(b)$. These structures also admit quotients by substructures, which can be thought of as identifying the elements of the substructure with the identity element of a new structure. Because of these similarities, we can use categorical language to describe algebraic structures in generality.

A *category* is a collection of objects and *morphisms*, which are maps between objects that may be composed [Rie14, Definition 1.1.1]. For example, **Ring** is the category whose objects are rings and whose morphisms are ring homomorphisms. A morphism is a *monomorphism* if it is left-cancellable, i.e. f is a monomorphism if for every morphisms g, h in its category, $f \circ g = f \circ h$ implies $g = h$ [Rie14, Definition 1.2.7]. These are a generalization of injective functions, and in some categories all monomorphisms are injections. Monomorphisms are represented by $\hookrightarrow$. Similarly, *epimorphisms* are right-cancellable morphisms, often coincide with surjections, and are represented by $\twoheadrightarrow$. The set of morphisms between an object A and an object B in a fixed category is $\operatorname{Hom}(A, B)$, and if the category is an *abelian category* then $\operatorname{Hom}(A, B)$ is an abelian group whose group operation is function composition.

Relationships between objects in categories are usually represented by diagrams. A diagram is commutative if any path taken along arrows (representing morphisms) between the same starting and ending objects yields an equality of compositions of those morphisms. An object Q is an *injective object* in a category if for any objects X, Y, monomorphism $f : X \to Y$, and morphism $g : X \to Q$, there exists a morphism h such that the diagram



commutes. This is equivalent to saying $\operatorname{Hom}(-, Q)$ maps monomorphisms to epimorphisms [Mac98, V.4].

A *sequence* is a one-dimensional diagram such as

$$0 \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \to 0$$

and is called *exact* at A, for example, if $g \circ f$ is the zero map, where f maps into A and g maps out of A. If a sequence is exact at each of its objects, it is called an exact sequence, and if it has the form above, it is called a *short exact sequence* [DF04, Definition E.5.1]. In categories that admit quotient objects, a short exact sequence like the one above indicates that $C \cong {}^{B}\!/_{A}$. If $B \cong A \oplus C$, it is called a *split exact sequence*.

Aside from diagrams representing relationships within a category, relationships between categories themselves can be described by *functors*. A *covariant* functor $F : \mathbf{C} \to \mathbf{D}$ associates to each object in $\mathbf{C}$ an object in $\mathbf{D}$, and to each morphism in $\mathbf{C}$ a morphism in $\mathbf{D}$. F is additionally required to respect composition of morphisms in $\mathbf{C}$ and to associate the identity morphism of any object C in $\mathbf{C}$ with the identity morphism in $\mathbf{D}$ of $F(C)$ [Rie14, Definition 1.3.1]. A covariant functor will often be referred to simply as a "functor." Because functors respect the composition of morphisms, a commutative diagram in $\mathbf{C}$ will induce a commutative diagram in $\mathbf{D}$. A *contravariant* functor operates similarly, with the key difference that the domain and codomain of a morphism are reversed under the action of the functor. Contravariant functors also carry diagrams from one category to another, with the caveat that the arrows will be reversed [Rie14, Definition 1.3.5].

With this in mind, in some category $\mathbf{C}$, if we fix an object A, then $\text{Hom}(A, -)$ is a covariant functor $\mathbf{C} \to \mathbf{Set}$ (where $\mathbf{Set}$ is the category whose objects are sets and whose morphisms are set inclusions). Likewise, $\text{Hom}(-, A)$ is a contravariant functor $\mathbf{C} \to \mathbf{Set}$. In abelian categories (which, fortunately, all of the ones relevant to us are), these are both functors $\mathbf{C} \to \mathbf{Ab}$ [Mac98, II.2]. For a functor $F : \mathbf{C} \to \mathbf{D}$ between abelian categories and a short exact sequence

$$0 \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} C \to 0$$

in $\mathbf{C}$, consider the sequence

$$0 \xrightarrow{F(f)} F(A) \xrightarrow{F(g)} F(B) \xrightarrow{F(h)} F(C) \to 0$$

While this sequence certainly exists, it may not be exact. However, if it is exact at $F(f)$ and $F(g)$ (i.e. it is a *left exact sequence*) then we call F a *left exact functor*; similarly, if the sequence is exact at $F(g)$ and $F(h)$ (a *right exact sequence*) then F is a *right exact functor*. A functor that is both left and right exact (i.e. it preserves exact sequences) is called an *exact functor* [Mac98, VIII.3]. It can be shown that $\text{Hom}(A, -)$ is left exact and $\text{Hom}(-, A)$ is right exact. Such an object A is called *projective* if $\text{Hom}(A, -)$ is right exact as well, and called *injective* if $\text{Hom}(-, A)$ is left exact.

1.2 Harmonic Analysis Preliminaries

Let

$$\int_U f(x)dm$$

denote the Lebesgue integral of a function $f : U \to \mathbb{C}$ with respect to the Lebesgue measure m [Rud87, See Theorem 2.20]. A function f is Lebesgue square-integrable if

$$\left(\int_U |f|^2 dm\right)^{\frac{1}{2}}$$

is finite. This quantity is the L^2-*norm* of f on U. $L^2(\mathbb{R})$ is the set of complex-valued Lebesgue square-integrable functions with domains in $\mathbb{R}$, and $L^1(\mathbb{R})$ is the set of functions f such that $|f|$ is Lebesgue-integrable [Rud87, Definition 3.6]. $L^2(\mathbb{R})$ is algebraically an infinite-dimensional inner product vector space and topologically a complete metric space with respect to the above norm [Rud87, Theorem 3.11]. The *Fourier transform* of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is

$$\widehat{f}(\zeta) = \int_{\mathbb{R}} f(x)e^{-2\pi i\zeta x}dm$$

Since $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, this formula can be extended by continuity to hold for all of $L^2(\mathbb{R})$ [PW12, Lemma 8.50]. The Fourier transform is an involution of $L^2(\mathbb{R})$ functions almost everywhere (i.e. everywhere except on a set of measure zero). The *inverse Fourier transform* allows us to recover f from $\widehat{f}$:

$$f(x) = \int_{\mathbb{R}} \widehat{f}(\zeta)e^{2\pi ix\zeta}dm(\zeta)$$

1.1 Remark. Operations on a function $f \in L^2(\mathbb{R})$ correspond to operations on $\widehat{f}$. For instance, the *translation* $f(x-h)$ corresponds to a *modulation*, i.e. $(f(x-h))\widehat{\ } = e^{-2\pi ih\zeta}\widehat{f}(\zeta)$.

In the twentieth century, Lev Pontryagin generalized the theory of the Fourier transform to **LCA** groups in the following way. In any topological space, a set is a *Borel set* if it is the countable union, countable intersection, or complement of open sets, including combinations of the above. For every locally compact abelian group G, there exists a measure μ that assigns a real number to each Borel set in G, a *Haar measure* on G, which is translation invariant, i.e. $\mu(gU) = \mu(U)$ for every $g \in G$, among some other nice properties [Rud90, 1.1.1]. Note that the Lebesgue measure on $\mathbb{R}$ is also translation invariant. The measurable subsets of G are precisely the elements of its Borel $\sigma-$algebra.

1.2 Remark. People often refer to "the" Haar measure on G – that is, the unique Haar measure such that $\mu(G) = 1$ [Rud90, 1.1.3]. However, in this text we often instead select the unique Haar measure normalized to have $\mu(H) = 1$ for a distinguished subgroup H. Following [BB04, p. 425], we also specify that the counting measure is to be used for discrete groups.

A *character* of G is a continuous group homomorphism $\chi : G \to \mathbb{T}$, where $\mathbb{T}$ denotes the circle group, the compact subgroup of $\mathbb{C}$ of complex numbers with modulus 1. The characters of G form an **LCA** group, denoted $\widehat{G}$ [Rud90, Theorem 1.2.6]. For example,

$$\begin{aligned} \widehat{(\mathbb{R},+)} \cong (\mathbb{R},+), \chi(x) &\leftrightarrow e^{iyx} \\ \widehat{\mathbb{T}} \cong \mathbb{Z}, \chi(x) &\leftrightarrow e^{2\pi inx} \end{aligned}$$

(for some $y \in \mathbb{R}$ and $n \in \mathbb{Z}$) are the group dualities relevant to classical Fourier analysis [Rud90, Example 1.2.7]. $L^2(G)$ for an **LCA** group G is defined to be the space of functions $G \to \mathbb{C}$ that are Lebesgue square-integrable with respect to the Haar measure on G.

1.3 Definition. The *Fourier transform* of a function $f \in L^2(G)$ is a function $\widehat{f} \in L^2(\widehat{G})$ defined by

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}d\mu$$

▽

This shares nearly all the nice properties of the Fourier transform on $\mathbb{R}$, including invertibility. [Rud90, 1.2.3, 1.2.4, 1.5.1].

1.3 Classical Multiresolution Analysis

Fourier analysis is not well-suited to every application, however. If $f : \mathbb{R} \to \mathbb{R}$ represents the amplitude of a song at a given time, to summarize a common example given in [PW12, 9.1], $\widehat{f}(\zeta)$ would represent the amplitude of only a specific given frequency throughout the entire song. In practice, since only a finite number of frequencies can be computed, $\widehat{f}$ cannot represent the full spectrum of frequencies present, and some information is lost. If we were to choose the frequencies that are to be recorded, we could easily end up with a choice that suits one segment of the song but not others – the dominant frequencies of a vocal interlude, for instance, will be substantially different from those of a drum roll. To solve this, frequency analysis of audio is usually done on individual subsets of the track, identifying the frequencies present in each $1-$millisecond time span, possibly. However, for some songs, there will be dramatically different frequencies present in adjacent subsets, causing audible discontinuities.

This problem is solved by decomposition of f not into a combination of sinusoids but a combination of transformations of a different function called a *wavelet*.

1.4 Definition. A wavelet is classically defined to be a function $\psi(x) \in L^2(\mathbb{R})$ such that $\left\{\psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k) \,\middle|\, j, k \in \mathbb{Z}\right\}$ is an orthonormal basis for $L^2(\mathbb{R})$ [PW12, Definition 9.15]. ▽

Like the Fourier transform, we define the *wavelet coefficients* of a function $f \in L^2(\mathbb{R})$ to be

$$Wf(j,k) = \int_{\mathbb{R}} f(x)\overline{\psi_{j,k}(x)}dx$$

which assigns a coefficient to f depending on the dilation factor j and the translation k [PW12, Definition 9.18]. Then, a reconstruction formula holds [PW12, Proposition 9.17]:

$$f(x) = \sum_{j,k\in\mathbb{Z}} W_f(j,k)\psi_{j,k}(x)$$

Fixing j in this formula yields a projection of f into a subspace of $L^2(\mathbb{R})$, which is in many cases the orthogonal complement of one subspace in a chain of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ satisfying certain conditions:

1.5 Definition. From [PW12, Definition 10.3], an *orthogonal multiresolution analysis* is a collection of closed subspaces V_j of $L^2(\mathbb{R})$ that satisfies the following properties for all $j \in \mathbb{Z}$:

(1) $V_j \subset V_{j+1}$

(2) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(3) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

(4) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$

(5) there exists a *scaling function* $\varphi(x) \in V_0$ such that $\{\varphi(x-k) \mid k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

∇

1.6 Remark. Some authors refer to the 0−scale wavelet ψ as the "mother wavelet."

The associated *wavelet spaces* are a collection of subspaces W_j such that $V_{j+1} = V_j \oplus W_j$. Just as each V_j is generated by translates of the scaling function φ, each W_j is generated by translates of a wavelet ψ. Thus, by projecting f into V_{j-1}, we lose the detail contained in W_{j-1}. Although the wavelet transform is lossless in theory, this lossy feature is one of the most widespread applications of wavelet transforms; by carefully selecting the wavelets we use, we can use an orthogonal projection to remove undesirable details while retaining desirable ones. In the previous analogy, a wavelet space W_j could be made to contain most of the background noise of an audio track, so projection of the signal into V_j could eliminate that noise.

Aside from trivial cases, it can be difficult to find wavelet functions, so as a result, it is common to begin with a scaling function as in (5) above. The existence of wavelets in such cases follows from the following theorem.

1.7 Theorem (Mallat)**.** For an MRA defined as above with scaling function φ, there exists a wavelet $\psi \in L^2(\mathbb{R})$ such that for each $j \in \mathbb{Z}$, $\{\psi_{j,k} \mid k \in \mathbb{Z}\}$ is an orthonormal basis for the wavelet space W_j [PW12, Theorem 10.18].

Usefully, for a scaling function φ satisfying the criteria of Mallat's theorem, the associated wavelet ψ can be computed. If we limit ourselves to a finite chain of subspaces, as we would in a signal processing application, there is an algorithm that can compute ψ in linear time. Later on, we will see that it is a central goal of abstract multiresolution analysis to find existence criteria for wavelet sets, as well as algorithms to find them in the absence of orthogonality.

Some of these definitional requirements are flexible – it is common to not require orthogonality for (5), as well as to require that each V_j and W_j are orthogonal. Additionally, the scaling operator $f(x) \mapsto f(2x)$ in (4) and the translation operators $\varphi(x) \mapsto \varphi(x-k)$ in (5) can be modified to better suit various applications. Constructions like "ridgelets" [CD00] and "shearlets" [KL12] have been posited to provide better approximations for certain signals than wavelets, which struggle with some types of discontinuity. With this in mind, it has been productive to develop a theory of multiresolution analysis in more generality. Before we set out on that expedition, however, we will review an example in the classical situation that will inform our considerations later on.

1.8 Example (Classical Haar MRA)**.** The Haar MRA is determined by the scaling function $\varphi(x) = \mathbb{1}_{[0,1)}(x)$. We define each subspace V_j to be the closure of the vector space with basis $\left\{\varphi_{j,k} \,\middle|\, k \in \mathbb{Z}\right\}$, which can be shown to be orthonormal for $j = 0$ (and indeed for every j). Each V_j consists of step functions constant on $[2^{-j}k, 2^{-j}(k+1))$ for each $k \in \mathbb{Z}$ – as j increases, the step functions are permitted to change more rapidly, increasing the "resolution." The step functions in W_j are constant on $[2^{-(j+1)}k, 2^{-(j+1)}(k+1))$ since $W_j \subset V_{j+1}$ and, in order to maintain orthogonality, must have an average value of 0 on $[2^{-j}k, 2^{-j}(k+1))$.

By Theorem 1.7, we know that we can find a wavelet ψ consistent with this system. Since $\varphi \in V_0$, we can compute the following for any $f \in V_0$:

$$\operatorname{orth}_{V_{-1}} f = \sum_{k \in \mathbb{Z}} \left(2^{\frac{-1}{2}} \int_{\mathbb{R}} f(x) \varphi\left(\frac{1}{2}x - k\right) dx \right) \varphi\left(\frac{1}{2}x - k\right)$$

and

$$\operatorname{orth}_{W_{-1}} f = f - \operatorname{orth}_{V_{-1}} f$$

Using the algorithm given by Mallat's theorem, we can show that our wavelet is in fact $\psi(x) = \mathbb{1}_{[\frac{1}{2},1)}(x) - \mathbb{1}_{[0,\frac{1}{2})}$ and can show that orthogonal projection onto W_{-1} coincides with projection onto the subspace generated by $\left\{\psi_{0,k} \,\middle|\, k \in \mathbb{Z}\right\}$. The full calculation can be seen in [PW12, 9.35]. $\Diamond$

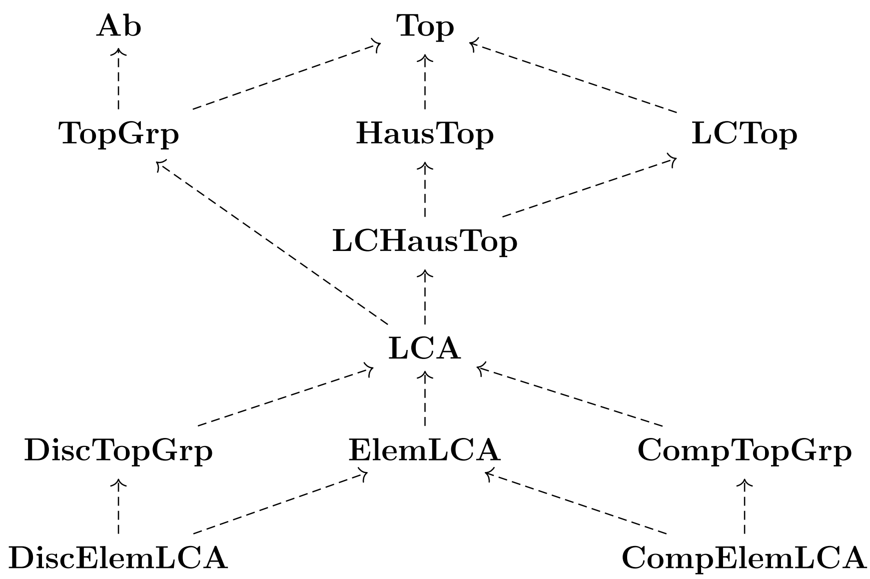
1.9 Remark. In the main body of this text, we will use v_0 instead of φ to denote the scaling function of a multiresolution analysis.

2 Scaling and Translation

Having established some of our goals in developing an abstract multiresolution analysis, we can now begin developing those tools. However, unlike in $L^2(\mathbb{R})$, it is unclear what "scaling" or "translation" might mean in an arbitrary group. In order to motivate potential definitions, we will first investigate a category-theoretic formulation of Pontryagin duality. We will then use this to illuminate a connection between Fourier theory and the lattices required for defining useful scaling and translation operators.

2.1 Duality

Let **Ab** be the category of abelian groups, **Top** be the category of topological spaces, **TopGrp** be the category of topological groups, **HausTop** be the category of Hausdorff topological spaces, **LCTop** be the category of locally-compact topological spaces, and **LCA** be the category of locally-compact abelian groups. The rest of the terminology is inspired by [Roe74], where an "elementary group" is a group of the form $\mathbb{T}^i \oplus \mathbb{Z}^j \oplus \mathbb{R}^k \oplus G$, with G a finite abelian group. We have the following diagram of categorical inclusion:



From this, we can define Pontryagin duality in a categorical sense.

2.1 Proposition. Let G and $f : G \to H$ denote an object and a morphism respectively in the category of topological groups. Define the dual group functor $\widehat{(-)}$: **TopGrp** $\to$ **TopGrp** by $G \mapsto \mathrm{Hom}(G, \mathbb{T})$ and $f \mapsto (\alpha \mapsto \alpha \circ f)$ for $\alpha \in \mathrm{Hom}(H, \mathbb{T})$. Then $\widehat{(-)}$ is contravariant and is exact on open subgroups included via open maps.

Proof. Contravariance is inherited from the $\mathrm{Hom}(-, \mathbb{T})$ functor. The additive group of $\mathbb{R}$ is a divisible group, so $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is as well; i.e. $\mathbb{T}$ is injective in **Ab**. Suppose that $0 \to$

$H \xrightarrow{a} G \xrightarrow{b} I \to 0$ is a short exact sequence of abelian groups. The functor $\mathrm{Hom}(-, \mathbb{T})$ is left-exact [Rie14, 4.5.11]. We then immediately have the exactness of $0 \to \widehat{I} \xrightarrow{\beta} \widehat{G} \xrightarrow{\alpha} \widehat{H}$, where $\alpha = \widehat{a}$ and $\beta = \widehat{b}$. Since $\mathbb{T}$ is injective in the category of topological groups and a is a monomorphism, for every $\chi \in \widehat{H}$ there exists a ξ such that the following diagram commutes:

$$\begin{array}{ccc} H & \overset{a}{\hookrightarrow} & G \\ \downarrow{\scriptstyle\chi} & \swarrow{\scriptstyle\xi} & \\ \mathbb{T} & & \end{array}$$

Since $\xi \in \widehat{G}$ and $\alpha(\xi) = \xi \circ a = \chi$, we have that α is an epimorphism in **Ab** but not necessarily in **TopGrp**, because it may not be continuous. If we suppose that our original sequence was exact in **TopGrp**, then a is by definition continuous. For ξ to be continuous, we would need χ to be continuous, so $\chi^{-1}(U)$ must be open in H for every open $U \subseteq \mathbb{T}$. This is satisfied by requiring that H is an open subgroup and a be an open map. In this case, α is a surjective continuous group homomorphism, so we may say that $0 \to \widehat{I} \xrightarrow{\beta} \widehat{G} \xrightarrow{\alpha} \widehat{H} \to 0$ is a short exact sequence of topological groups. □

With the small amount of machinery we have so far developed, the following result is a simple exercise in general topology [Roe74, Proposition 2].

2.2 Proposition. Let G be an LCA group. If G is compact then $\widehat{G}$ is discrete, and if G is discrete then $\widehat{G}$ is compact.

Recalling Definition 1.3, the inversion formula following from this duality is a classical result:

2.3 Definition. The *inverse Fourier transform* of a function $f \in L^2(\widehat{G})$ is

$$\check{f}(x) = \int_{\widehat{G}} f(\chi)\chi(x)d\nu(\chi)$$

where ν is the corresponding normalized Haar measure on $\widehat{G}$. ▽

As mentioned in Remark 1.2, although both the Fourier transform and its inverse depend on a choice of normalization for the Haar measure, we can require that the measure chosen in $\widehat{G}$ is normalized such that the inversion property holds.

Translation of a multiresolution analysis on L^2 of a group G is obtained via a *lattice* on G. For us, a lattice L is a discrete subgroup of an **LCA** group G such that the quotient G/L is compact. The canonical short exact sequence

$$L \hookrightarrow G \twoheadrightarrow {}^{G}\!/_{L}$$

induces the dual sequence

$$\widehat{{}^{G}\!/_{L}} \hookrightarrow \widehat{G} \twoheadrightarrow \widehat{L}$$

and since L is discrete, $\widehat{L}$ is compact. We then define the *dual lattice* of L to be $L^* = \widehat{G/L}$. In this case, given $f \in L^2(G)$, its translation by $h \in L$, $\tau_h f$, is defined by $f(hx)$.

In general, however, not every **LCA** group has a discrete subgroup that can serve as a lattice. It is possible, then, to define translation with respect to a compact open subgroup H instead. We define H^* identically, that is, $H^* = \widehat{G/H}$, but H^* is again open and compact. Note that for any subgroup H, H^* is compact if and only if H is open, and H is compact if and only if H^* is open [BB04, Section 1.3].

2.2 Expansive automorphisms

Scaling in the classical setting is usually done by an *expansive* matrix (i.e. all eigenvalues are greater than one in absolute value). We will develop a scaling operator as an element of $\mathrm{Aut}(G)$.

Similar to the lattice, an automorphism δ admits an *adjoint* automorphism δ^* on $\widehat{G}$. It follows that the image of a character χ under an adjoint automorphism δ^* can be characterized by $\delta^*(\chi) = (x \mapsto \chi(\delta(x)))$.

2.4 Definition. Let a be an automorphism of G and μ be a Haar measure on G. The *modulus* of a is the quantity $|a|$ such that $\mu(a(U)) = |a|\mu(U)$ for any nonempty measurable $U \subseteq G$. ▽

Expansiveness in arbitrary **LCA** groups has been defined in several ways, and is not required by every author in the construction of an MRA, but we will define it according to [BB04, 2.5].

2.5 Definition. An automorphism δ of an **LCA** group G is *expansive* with respect to a compact open subgroup H if

$$H \subsetneq \delta(H) \tag{1}$$

$$\text{and} \bigcap_{n\geq 0} \delta^{-n}(H) = \{0\} \tag{2}$$

▽

2.6 Proposition. If δ is an expansive automorphism then $|\delta| > 1$.

Proof. Since $H \subsetneq \delta(H)$, for every $s \in \delta(H)$ there is a corresponding coset $s + H \in \delta(H)/H$. Because $\delta(H)/H$ is a discrete group we use the counting measure (see Remark 1.2), so $\mu(s + H) = 1$ and

$$\mu(\delta(H)) = \sum_{s\in\delta(H)} \mu(s + H) = \left|\delta(H)/H\right|$$

which is greater than 1. □

Definition 2.5 can be equivalently expressed in terms of adjoints:

2.7 Lemma. An automorphism δ of an **LCA** group G is expansive with respect to a compact open subgroup H if and only if

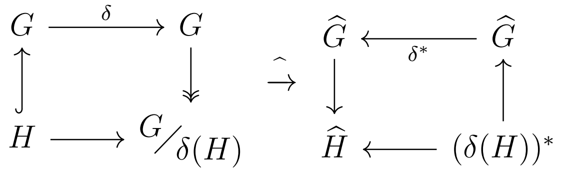
$$H^* \subsetneq \delta^*(H^*) \tag{3}$$

$$\text{and } \bigcup_{j\geq 0} \delta^{*^j}(H^*) = \widehat{G} \tag{4}$$

Proof. First, to show that (1) $\Rightarrow$ (3), suppose that δ is expansive, let $\chi \in H^*$, and $x \in H$. (1) implies $x \in \delta(H)$, so let y be the preimage of x in H. Considering the adjoint automorphism δ^* on $\widehat{G}$, the preimage $\delta^{*^{-1}}(\chi)$ is again an element of $\widehat{G}$. We then calculate $\delta^{*^{-1}}(\chi)(x) = \delta^{*^{-1}}(\chi)(\delta(y)) = \delta^{-1^*}(\chi)(\delta(y)) = \chi(\delta^{-1}(\delta(y))) = \chi(y)$, but since $y \in H$, $\chi(y) = 1$, we have shown that $\delta^{*^{-1}}(\chi) \in H^*$. Furthermore, if $H^* = \delta^*(H^*)$, then $|\delta^*| = 1$ by definition, but $|\delta^*| = |\delta|$ [see BB04, Section 1.1], contradicting Proposition 2.6, so $H^* \subsetneq \delta^*(H^*)$.

Second, to show that (2) $\Rightarrow$ (4), observe that $\delta^*(H^*) \subseteq \widehat{G}$ by the definition of the adjoint. Let $\chi \in \widehat{G}$ and suppose the contrary, that $\chi \notin \bigcup_{j\geq 0}(\delta^*)^j(H^*)$. It follows that for every $j \geq 0$, we can find a $x_j \in (\delta^{j^{-1}})(H)$ such that $\chi(x_j) \neq 1$ (if $\chi(x_j) = 1$ for every x_j then χ would be in the adjoint of the subgroup $(\delta^{j^{-1}})(H)$ which is $(\delta^*)^j(H^*)$). Let y_j be some power of x_j such that $\mathrm{Re}(\chi(y_j)) < 0$. Since $y_j \in \delta^{j^{-1}}(H) \subseteq H$ and H is compact, it contains a limit point of the sequence (y_j) which we will denote y. By (2), there is an integer N such that $y \notin \delta^{N^{-1}}(H)$, so (1) implies that $y \notin \delta^{n^{-1}}(H)$ for all $n \geq N$. The coset $y + \delta^{-n}(H)$, considered by inclusion in H is open but contains no more than N elements of the y_j sequence, a contradiction unless $y = 0$. The set $U = \chi^{-1}((0,\infty))|_H$ is open in H by continuity, and contains 0 as the limit point of this sequence, but we defined the y_js to have strictly negative real part, so $y_j \notin U$ for any j, a contradiction. It follows that $\chi \in (\delta^*)^j(H^*)$ for some j. The converse of both properties can be shown similarly. $\square$

Thus, we can define expansive automorphisms either with respect to a compact open subgroup H or by the adjoint automorphism's action on the subgroup's adjoint. The equivalence of conditions (1) in Definition 2.5 and (3) in Lemma 2.7 is reflected by applying the dual functor to the following diagram.



With this in place, since the modulus of an expansive matrix in a classical MRA, like the modulus of our expansive automorphism δ, must be strictly greater than 1, we may feel comfortable with **LCA** groups having sufficient structure to perform multiresolution analysis.

3 Constructing the MRA

We will now define the dilation and translation operators of an MRA. The dilation operator depends on an automorphism (of topological groups) δ on G, but the translation operator is slightly more complex. Before attempting that task, we will try and capture the essence of what is needed from these operators.

3.1 In Full Generality

We start by taking a translation operator τ and a dilation operator σ, merely as symbols with no explicit meaning. We define the family of translation operators $\{\tau_j\}$ recursively by the relation $\tau_j = \sigma^j \tau \sigma^{-j}$ and a family of translation cyclic groups $T_j = \langle \tau_j \rangle$. The full translation group is $T = \cup_{j \geq 0} T_{-j}$. Having this family of groups is critical to the "multiresolution" part of multiresolution analysis. We can combine the translation and scaling operations by defining the full transformation group X to be a split extension of $\langle \sigma \rangle$ by T, i.e.

$$T \hookrightarrow X \twoheadrightarrow \langle \sigma \rangle$$

is a split exact sequence of groups.

With X representing all transformations we might apply to a function, we must now make reference to the function space on which we're defining the MRA. In the classical case, MRAs are defined as subspaces of $L^2(\mathbb{R})$, and more recent work on MRAs in other fields are still concerned with complex-valued function spaces, such as $L^2(\mathbb{Q}_p)$. However, we may consider a more general case.

3.1 Definition. Let k be any field. Considered as a collection of $kX-$modules, a *multiresolution analysis* with respect to v_0 is a set $\left\{ V_j = v_0(kT_j) \,\middle|\, j \in \mathbb{Z} \right\}$. $\triangledown$

In this situation, v_0 takes the place of the scaling function.

The usual method of defining MRAs, even generalized ones, is axiomatic. By taking a more constructive approach, we can capture the essential structure of an MRA free from dependence on a particular **LCA** group.

Our dilation operator will be an element of kX defined by $\sigma = (a \cdot -) \circ \delta$, so it acts upon elements of V_j first by the action of δ, then by post-multiplication by a fixed element $a \in k$.

3.2 Example (The Haar MRA, revisited)**.** In general, define the action of elements of the general affine group over $\mathbb{R}$ on a function $f : \mathbb{R} \to \mathbb{R}$ by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f(x) := f\left(\frac{x-b}{a}\right)$$

The Haar MRA has dilation and translation operators

$$\delta = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \tau_j = \begin{pmatrix} 1 & 2^j \\ 0 & 1 \end{pmatrix}$$

and is generated by taking v_0 to be the characteristic function of the unit interval. In this case, our lattice L is the discrete subgroup $\mathbb{Z}$. ◊

A result of our more general definition of multiresolution analyses is that they may not necessarily be infinite-dimensional, but the examples given in [BB04] almost universally are. Infinite-dimensional MRAs are "nice" algebraic objects as they correspond to the quotient of kX by a principal ideal.

Much of the current literature deals with real-valued functions, taking v_0 to be an element of $L^2(G)$ and, accordingly, $k = \mathbb{R}$. In this case, the dilation operator is usually defined by $\sigma(f) = |\delta|^{\frac{1}{2}}(f \circ \delta)$.

3.2 Translation Troubles

Returning to translation, if a discrete subgroup L is available to serve as a lattice, the naïve choice of translation operator, i.e. $g \mapsto s + g$ for some $s \in L$ can be used. However, if we are defining translation with respect to a compact open subgroup H instead, we must be more careful. [BB04] suggests an alternate translation operator that may be applicable in both situations:

Let H be a subgroup of an **LCA** group G. For each $r + H^* \in \widehat{H}$, fix $\varphi : \widehat{H} \to \widehat{G}$ as an injective right inverse of the quotient map $\widehat{G} \to \widehat{H}$, i.e. φ maps each distinct coset $r + H^*$ to an arbitrarily-chosen representative $r \in \widehat{G}$. Define $\mathcal{D} = \varphi(\widehat{H})$. In general $\mathcal{D}$ is not a subgroup of $\widehat{G}$. We then further define $\theta : \widehat{G} \to \mathcal{D}$ by

$$\chi \mapsto \varphi(\chi + H^*)$$

and $\nu : \widehat{G} \to H^*$ by

$$\chi \mapsto \chi - \theta(\chi)$$

For $s + H$ in G/H define $w_{[s]} : \widehat{G} \to \mathbb{C}$ by

$$\chi \mapsto \overline{(\nu(\chi))(s)}$$

and $m_{[s]} : L^2(\widehat{G}) \to L^2(\widehat{G})$ by

$$f(\chi) \mapsto w_{[s]}(\chi) f(\chi)$$

Then the translation-by-s operator acting on an element f of $L^2(G)$ is defined by

$$\tau_{[s]} f = f * (w_{[s]})^{\smile} = \left(m_{[s]}\widehat{f}\right)^{\smile} \tag{5}$$

[BB04, Definition 2.1]. $m_{[s]}$ is a modulation operator analogous to the classical Fourier modulation in Remark 1.1.

3.3 Example (Classical Fourier translation)**.** Consider $\mathbb{T}$ as a compact open subgroup of $\mathbb{R}$. It is well-known that $\widehat{\mathbb{T}} = \mathbb{Z}$, so in the notation above, we have

$$\begin{aligned} G &= \mathbb{R} \\ \widehat{G} &= \mathbb{R}, (x \mapsto e^{2\pi i x}) \leftrightarrow x \\ H &= \mathbb{T} \\ \widehat{H} &= \mathbb{Z}, (e^{2\pi i n x} \mapsto e^{2\pi i n x}) \leftrightarrow n \\ H^* &= \mathbb{T}, (\left(e^{2\pi i x}\right)^n) \leftrightarrow e^{2\pi i x} \end{aligned}$$

with the maps

$$\begin{aligned} \varphi((e^{2\pi i n x} \mapsto e^{2\pi i n x})) &= (e^{2\pi i n x} \mapsto e^{2\pi i x}) \\ \theta(\zeta) &= \varphi(e^{2\pi i x \zeta}) \\ \nu((x \mapsto e^{2\pi i n x})) &= e^{-2\pi i x} \\ w_{[s]}((x \mapsto e^{2\pi i n x})) &= e^{2\pi i s} \\ m_{[s]}(f(\zeta)) &= e^{2\pi i s \zeta} f(\zeta) \\ \tau_{[s]}(f(x)) &= \left(\zeta \mapsto e^{-2\pi i s \zeta} \widehat{f}(\zeta)\right)^{\smile} \end{aligned}$$

Expanding this, we have

$$\tau_{[s]}(f(x)) = \left(\int_{\mathbb{R}} e^{-2\pi i \zeta(x+s)} f(x) dx\right)^{\smile}(x)$$

which resolves to the traditional translation operator in $L^2(\mathbb{R})$, given by

$$\tau_{[s]} f(x) = f(x - s)$$

As in Remark 1.1, a translated function is the inverse Fourier transform of a modulated function. ◊

While this may appear convoluted, this translation operator does not depend on the presence of a discrete subgroup, but coincides naturally with translation operators determined by such a subgroup when one exists.

3.4 Example (Translations on $\mathbb{F}_p((t))$)**.** This example elaborates upon [BB04, Example 2.11; Ben04, Example 3.4]. We take G to be the group $\mathbb{F}_p((t))$ of formal Laurent series with coefficients in the field of p elements for some prime p, and our compact open subgroup H will be the set $\mathbb{F}_p[[t]]$ of formal power series. The quotient $\mathbb{F}_p((t)) / \mathbb{F}_p[[t]]$ contains Laurent series with terms of strictly negative degree and is therefore isomorphic to $\mathbb{F}_p[t]$, the polynomial ring, which is itself a discrete subgroup.

If we define $\chi : G \to \mathbb{C}$ by

$$\chi\left(\sum_{n \geq n_0} c_n t^n\right) = e^{2\pi i c_{-1}/p}$$

it can be shown that every $\gamma \in \widehat{G}$ has the form $\gamma(x) = \chi(\alpha x)$ for some $\alpha \in G$, so G is self-dual. It follows that $H^* \cong \mathbb{F}_p[[t]]$ as well. We will identify the character γ with the group element α.

We choose the automorphism $\delta(x) = t^{-1}x$ and the map $\varphi : \widehat{H} \to \widehat{G}$ to be

$$\varphi\left(\left(\sum_{n\geq n_0} c_n t^n\right) + H^*\right) = \sum_{n=n_0}^{-1} c_n t^n$$

essentially discarding nonnegative degree terms. Suppose $\chi(\alpha x) = \gamma(x)$ and let c_i denote the coefficients of α for a character γ below. Following Example 3.3, we have the induced maps

$$\theta : \widehat{G} \to \mathcal{D}, \sum_{n\geq n_0} c_n t^n \mapsto \sum_{n=n_0}^{-1} c_n t^n$$
$$\nu : \widehat{G} \to H^*, \sum_{n\geq n_0} c_n t^n \mapsto \sum_{n\geq 0} c_n t^n$$

We then calculate $w_{[s]}$ for some $s \in {}^{G}\!/\!_{H}$

$$\begin{aligned} w_{[s]} : \widehat{G} \to \mathbb{C}, \gamma &\mapsto \overline{(\nu(\gamma))\,(s)} \\ \alpha &\mapsto \overline{\chi\,(s\nu(\alpha))} \\ \sum_{n\geq n_0} c_n t^n &\mapsto \chi\left(-s\sum_{n\geq 0} c_n t^n\right) \end{aligned}$$

in order to obtain the modulation

$$\begin{aligned} m_{[s]} : \widehat{f}(\gamma) &\mapsto \overline{(\nu(\gamma))(s)}\widehat{f}(\gamma) \\ \widehat{f}(\alpha) &\mapsto \overline{(\chi(s\nu(\alpha)))(s)}\widehat{f}(\alpha) \end{aligned}$$

which induces the translation

$$\tau_{[s]} : f(x) \mapsto \left(\overline{\chi(s\nu(\alpha))}\widehat{f}(\alpha)\right)^{\smile}$$

$\Diamond$

4 Wavelet Sets

Current literature on wavelet sets deals primarily with characterizing the existence conditions of multiresolution analyses with respect to scaling functions of the form $v_0 = \widehat{\mathbb{1}}_E$ for some measurable $E \subseteq \widehat{G}$. [Spe03] investigates the case in $\mathbb{R}^n$ where dilation is given by an automorphism that may not be expansive, and [BB04] considers indicator functions on several other self-dual fields. However, in the construction of Eq. (5), it is not immediately clear what the action of $\tau_{[s]}$ on such an indicator function is.

4.1 Proposition. Fix an expansive automorphism δ, $\tau_{[s]}$ defined as in Eq. (5), some $c \in G$, and $r \in \mathbb{Z}$. If $r \leq 0$, let $\theta_0 = \theta(0)$; otherwise, let $\{\theta_i\}_{0 \leq i < |\delta|^r} = \mathcal{D} \cap (\delta^*)^r(H^*)$.

If $r \leq 0$ then

$$\left(\tau_{[s]}(\mathbb{1}_{c+\delta^{-r}H})\right)(x) = \theta_0(s)\mathbb{1}_{s+c+\delta^{-r}H}(x) \tag{6}$$

otherwise,

$$\left(\tau_{[s]}(\mathbb{1}_{c+\delta^{-r}H})\right)(x) = \frac{1}{|\delta|^r}\left(\sum_{i=0}^{|\delta|^r-1}\theta_i(x-c)\right)\mathbb{1}_{s+c+H}(x) \tag{7}$$

Proof sketch. Let $f = \mathbb{1}_{c+\delta^{-r}(H)}$. Benedetto first shows that $\widehat{f}(\gamma) = |\delta|^{-r}\overline{\gamma(c)}\mathbb{1}_{(\delta^*)^r(H^*)}(\gamma)$. By the definition in Eq. (5),

$$\tilde{\tau}_{[s]}(\widehat{f}(\gamma)) = |\delta|^{-r}\overline{(\nu(\gamma)(s))\gamma(c)}\mathbb{1}_{(\delta^*)^r(H^*)}(\gamma)$$

If $r \leq 0$ then $(\delta^*)^r(H^*) \subsetneq H^*$, so for the indicator function to be nonzero, $\gamma \in H^*$ which implies $\nu(\gamma) = \gamma - \theta_0$. It follows that

$$\begin{aligned}
\tilde{\tau}_{[s]}(\widehat{f}(\gamma)) &= |\delta|^{-r}\overline{\gamma(s) - \theta_0(s)\gamma(c)}\mathbb{1}_{(\delta^*)^r(H^*)}(\gamma) \\
&= |\delta|^{-r}\theta_0(s)\overline{\gamma(s+c)}\mathbb{1}_{(\delta^*)^r(H^*)}(\gamma) \\
\tau_{[s]}(f(x)) &= |\delta|^{-r}\theta_0(s)\int_{H^*}\gamma(x)\overline{\gamma(s+c)}\mathbb{1}_{(\delta^*)^r(H^*)}(\gamma)d\mu(\gamma) \\
&= |\delta|^{-r}\theta_0(s)\int_{(\delta^*)^r(H^*)}\gamma(x)\overline{\gamma(s+c)}d\mu(\gamma) \\
&= \theta_0(s)\mathbb{1}_{s+c+\delta^{-r}(H)}(x)
\end{aligned}$$

The case where $r > 0$ is similar. □

4.2 Example (Haar wavelet on $\mathbb{F}_p((t))$)**.** Continuing Example 3.4, if $\delta(x) = t^{-1}x$, we have that $|\delta| = \mu(t^{-1}H) = p$, so Eq. (7) gives that

$$(\tau_{[s]}\mathbb{1}_{c+(t^rH)})(x) = p^{-r}\left(\sum_{i=1}^{p^r-1}\theta_i(x-c)\right)\mathbb{1}_{s+c+H}(x)$$

Recall $\theta_i(-) = \chi(\alpha_i -)$ for some $\alpha_i \in \mathcal{D} \cap t^{-r}H = \mathcal{D} \cap t^{-r}\mathbb{F}_p[[t]]$, and we chose φ such that every element of $\mathcal{D}$ has only terms of negative degree. Further, by definition, $\theta_i(x-c) = 1 \Leftrightarrow x - c \in H^* = \mathbb{F}_p[[t]]$ so

$$\chi(\alpha_i(x-c))\mathbb{1}_{s+c+H}(x) = \chi(\alpha_i(x-c))\mathbb{1}_{s+H}(x-c)$$

is nonzero if and only if $x - c \in s + H$, i.e. $x - c \notin \mathbb{F}_p[[t]]$, so $\theta_i(x-c) = 1$ always and

$$\tau_{[s]}\mathbb{1}_{c+(t^r H)}(x) = p^{-r}(p^r)\mathbb{1}_{s+c+H}(x) = \mathbb{1}_{s+c+H}(x)$$

This again reproduces a translation operator very similar to the usual translation operator on $\mathbb{R}$. ◊

In $\mathbb{R}^n$, the appropriate choice of dilation operator δ and translation lattice L guarantees the existence of a measurable $E \subseteq \mathbb{R}^n$ whose dilations tile $\mathbb{R}^n$ and translations pack $\mathbb{R}^n$ almost everywhere. We call E a *wavelet set* in this case. This is sufficient to be able to take $\mathbb{1}_E$ to be the scaling function of a Haar MRA [BS21, 2.5]. The Eq. (5) translation operator has a similar existence criteria for wavelet sets.

4.3 Theorem (From [BB04, Theorem 3.5])**.** Let G be an **LCA** group, H a compact open subgroup, $\tau_{[s]}$ be the translation operator defined in Eq. (5), and Δ be a nonempty countable subset of $\mathrm{Aut}(G)$. Then a measurable $E \subseteq \widehat{G}$ is a wavelet set if and only if

(a) $\{\delta^* E \mid \delta \in \Delta\}$ tiles $\widehat{G}$ almost everywhere; and

(b) there is a countable partition E_i of E into measurable subsets and a $\gamma_i, \gamma_i' \in \mathcal{D}$ for each E_i such that $E_i \subseteq \gamma_i + H^*$ and $E_i + \gamma_i'$ partitions H^* almost everywhere.

The existence of such a wavelet set E immediately lets us take $(\mathbb{1}_E)^{\vee}$ to be our wavelet.

[Foo05, Section 5] offers a module-theoretic criterion with the notation and less stringent requirements than Theorem 4.3:

4.4 Definition. An infinite-dimensional multiresolution analysis $v_0(kX)$ has a *wavelet basis* if V_{-1} is a pure $kT_{-1}-$submodule of V_0. A wavelet for this MRA is then any $\psi \in V_0$ such that $\psi(kT_{-1}) \oplus v_0(kT_{-1}) = V_0$. ▽

While v_0 generates V_0 as a kT_0-module, $\{\sigma^{-1}v_0, \tau_{-1}\sigma^{-1}v_0\}$ generate V_0 as a $kT_{-1}-$module.

4.5 Remark. [Foo05] uses the convention, favored by some authors, that $V_0 \oplus W_0 = V_{-1}$. We are representing his results using our convention that $V_{-1} \oplus W_{-1} = V_0$ for consistency.

Here, we can see that the lack of orthogonality causes Theorem 1.7 to not necessarily hold. However, for any kX, we can construct a scaling function v_0 such that $v_0(kX)$ has a wavelet basis.

By induction, then $V_{-i} = V_{-(i+1)} \oplus W_{-(i+1)}$ where $W_{-(i+1)}$ is a cyclic $kT_{-(i+1)}-$module. This gives us the familiar structure of the classical MRA, where each $W_{-(i+1)}$ can be considered a "detail subspace" of V_{-i}.

Because $v_0(kX)$ is infinite, it is isomorphic to the quotient of kX by a principal ideal so we can write a scaling equation using the scaling function: $v_0 = \delta^{-1} f(\tau_0) v_0$ for some $f \in k[X]$ [Foo05, 4.1]. We then decompose f into even and odd powers, i.e.

$$f(X) = X^{2n_1+1} f_1(X^2) + X^{2n_2} f_2(X^2)$$

for some $f_1, f_2 \in F[X]$. While this appears bizarre, it yields a representation of the scaling function in terms of the generators of V_0 as a $kT_{-1}-$module:

$$v_0 = \left(\tau_0^{n_1} \tau_{-1} f_1(\tau_0) \delta^{-1} + \tau_0^{n_2} f_2(\tau_0) \delta^{-1}\right) v_0 \tag{8}$$

This form allows us to use the following criterion:

4.6 Theorem. $v_0(kX)$ has a wavelet basis if and only if either

(a) $f_1 = 0$ and f_2 is a nonzero constant (or vice versa); or

(b) f_1 and f_2 are relatively prime.

(see [Foo05, 5.1]).

4.7 Example (Visiting the Haar MRA one last time)**.** Recall from Example 1.8 that the Haar MRA on $L^2(\mathbb{R})$ has scaling function $\mathbb{1}_{[0,1)}$. We can then write the scaling equation as $\mathbb{1}_{[0,1)}(x) = \delta^{-1}(\tau_0 + 1)\mathbb{1}_{[0,1)}$ using δ and τ from Example 3.2. In the notation above, $f(X) = X + 1$ so $f_1(X) = 1$ and $f_2(X) = 0$ with $n_1 = n_2 = 0$. This satisfies condition (a) in Theorem 4.6, so $\mathbb{1}_{[0,1)}$ is a scaling function for which a wavelet exists.

Additionally, [Foo05, 5.4] allows us to calculate the wavelet associated with this scaling function, which in this case is $(\tau_{-1} - 1)\delta^{-1}\mathbb{1}_{[0,1)}$. It is a simple exercise to show that this coincides with the familiar Haar mother,

$$\begin{aligned} \psi(x) &= \mathbb{1}_{[0,1)}(2x - 1) - \mathbb{1}_{[0,1)}(2x) \\ &= \mathbb{1}_{[\frac{1}{2},1)}(x) - \mathbb{1}_{[0,\frac{1}{2})} \end{aligned}$$

◊

So far, we have shown several of the approaches commonly used to build MRA-like apparatus on nontraditional algebraic structures. Although MRAs with neither discrete lattices nor orthogonal subspaces can behave very similarly to the most structured ones over $L^2(\mathbb{R})$, it is still unclear if there is a criterion for the existence of wavelet sets in this generality, with new results pertaining to this question over spaces as well-studied as $L^2(\mathbb{R}^n)$ emerging recently as in [BS21]. Further work might include investigating the algebraic properties of MRAs as $kX-$modules in the particular case where $k = L^2(G)$ for an **LCA** group G, as well as constructing non-Haar wavelets for such a situation.

Appendix: Notation Reference

Throughout this text, we will fix some notation. The sets in the list below may carry additional algebraic structure depending on context.

$\mathbb{R}$	The set of real numbers
$\mathbb{Z}$	The set of integers
$\mathbb{T}$	The circle group, also denoted S^1, consisting of complex numbers of the form $e^{2\pi ix}$ for some $x \in [0,1)$.
$\mathbb{1}_E$	The characteristic function of a set E, i.e. $\mathbb{1}_E(x)$ equals 1 if $x \in E$ and 0 otherwise.
$\mathbb{F}_p$	The finite field with p elements, where p is a prime number.
$F[x], F(x), F[[x]], F((x))$	For a field F, the rings of polynomials, rational functions, power series, and Laurent series respectively in a variable x.

Names of categories are typeset in bold. The following are some non-obvious ones:

Ab	abelian groups with group homomorphisms
TopGrp	Topological groups with continuous group homomorphisms.
R − Mod	Left modules over a ring R with $R-$module homomorphisms.
LCA	Locally-compact abelian groups with continuous group homomorphisms.

We use some symbols to usually represent certain objects, though their meanings are not fixed:

G, H	A group (or a subgroup)
R, S, A	A ring
I, J	A ring ideal
L	A discrete subgroup (a lattice)
k	A field
vR	A cyclic left $R-$module generated by v
φ^*, H^*	The dual or adjoint of a morphism φ or a subgroup H
x, y	An element of a group
ζ	The frequency-domain variable of classical Fourier analysis on $\mathbb{R}$.
χ, γ	An element of a dual group (i.e. a character), usually viewed as a map
μ	The unique normalized Haar measure on a given group
$d\mu(x), d\mu(\gamma)$	Differential used to indicate that a Lebesgue integral is with respect to the measure μ in the variable x or the variable γ

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